Kmth price sealed-bid auctions with general independent values and equilibrium linear mark-ups

Skitmore, Martin

Published in:
Journal of the Operational Research Society

DOI:
10.1057/jors.2013.163

km^{th} Price Sealed Bid Auctions with General Independent Values and Equilibrium Linear Mark-Ups

ABSTRACT

A generalised bidding model is developed to calculate the bidder's expected profit and auctioner's expected revenue/payment for both a General Independent Value (GIV) and Independent Private Value (IPV) km^{th} price sealed bid auction (where the m^{th} bidder wins at the k^{th} bid payment) using a linear (affine) mark-up function. The Common Value (CV) assumption, and *highbid* and *lowbid* symmetric and asymmetric First Price Auctions (FPA) and Second Price Auctions (SPA) are included as special cases. The optimal n bidder symmetric analytical results are then provided for the uniform IPV and CV models in equilibrium. Final comments concern implications, the assumptions involved and prospects for further research.

Key words: Auction Theory, General Independent Value Model, Mark up Pricing, km^{th} Price, Independent Private Values, Common Values.

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

INTRODUCTION

While the practice of auctioning goes back to ancient times[1], the earliest academic treatments are relatively recent, with the contributions of Friedman (1956) from an operations research (decision theoretic) perspective, Vickery (1961) from a game theoretic perspective and Gates (1967) from what has been termed the tendering theory perspective (Runeson and Skitmore 1999). All three approaches have some impractical assumptions. Decision theory (DT), for example, is essentially static, in that it assumes any given bidder's opponents to bid with either a random or constant mark-up. Game theory on the other hand assumes all bidders somehow always bid optimally as some, usually unstated, function of their item cost/value estimates.

Of the three, progress has been dominated by the development of the game theoretic approach into a full-blown Bayesian-Nash equilibrium theory, now termed Auction Theory (AT), under the standard economic assumption of rational utility maximisation – so that now "the auction problem can be understood by applying the usual logic of marginal revenue versus marginal cost" (Klemperer, 1999: 312)[2].

One of the major outcomes of this theoretical development has been to discover the equilibrium bidding strategies for independent *private value* (IPV) auctions. This assumes an idealised form of valuation process in which bidders privately know their own value[3] of the

---

[1] Cassady (1967) mentions a report by the Greek historian Herodotus, who described the sale by auction of women to be wives in Babylonia around the fifth century BC.

[2] The most notable contributions have come from Griesmer *et al* (1967), Wilson (1969, 1977), Milgrom (1979, 1981), Riley and Samuelson (1981), Myerson (1981) and Milgrom and Weber (1982) – see Klemperer (1999) for a comprehensive account.

[3] The term *value* is commonly used for *highbid* auctions. For *lowbid* auctions, such as occurs in tendering for construction work, the equivalent term is *cost* (Flanagan and Norman 1985). For *highbid* auctions, the bid denotes the amount of money paid by the bidder to the auctioner and vice versa for *lowbid* auctions. The amount of money changing hands is determined by the auction payment rule. For example, the first price auction (FPA) payment rule is that the amount of money changing hands is the amount of the winning bid – the highest bid in the case of *highbid* auction and the lowest (***n***th highest) bid in the case of *lowbid* auctions. For the second price auction (SPA), on the other hand, the auction payment rule is that the amount of money changing hands is the amount of second highest bid in *highbid* auctions and second lowest (*n*-1th highest) in *lowbid* auctions. Therefore, the bidder's profit is obtained by subtracting the payment from the bidder's value in *highbid* auctions and by subtracting the bidder's cost from the payment in *lowbid* auctions. As both *highbid* and

auctioned item perfectly accurately (but not the specific item values of opponents). For example, Vickrey (1961) showed that if bidders are symmetric (that is, their item values are drawn from the same probability distribution, the parameters of which are known to all bidders), the expected revenue to the auctioner in English first-price (open-cry), sealed-bid, second-price sealed-bid (Vickrey) and Dutch (descending) auctions is the same in equilibrium.

As an alternative to IPV auctions, in which each bidder is assumed to have different (perfectly estimated) item values, the *common value* (CV) model has been studied, in which the item value is assumed to be the same for all bidders, but imperfectly and privately estimated by each bidder (e.g., Wilson 1969)[4]. Clearly, the private and common value assumptions are special cases of a more general model in which bidders have *both* different item values *and* imperfect estimates of them. One version of this that has received considerable attention (e.g., Myerson, 1981; Riley and Samuelson, 1981; Milgrom and Weber, 1982) assumes that each bidder receives a private value estimate, but allows each bidder's item value to be a function of all the value estimates. With a suitable definition of this function in terms of the assumed conditional probabilities involved, Milgrom and Weber (1982) were able to develop the general model needed, termed the *affiliated values* model, by using a natural generalisation of the monotone likelihood ratio property commonly used in statistical models. This provides several equilibrium results, the most important of which is that the English auction generates the highest prices followed by the second-price and, finally, the Dutch and first-price auction.

Milgrom and Weber's work, however, is concerned with the general properties of symmetric auction models when value estimates are *not* independently distributed (Monteiro

---

lowbid auctions are dealt with simultaneously in this paper, the term 'item value' is used throughout to denote value/cost and the term 'payment' to denote the amount of money changing hands.

[4] The (perfectly) estimated value for IPV auctions is termed the 'type' while the (imperfectly) estimated value for CV auctions is termed the 'signal'. Here we use the term 'value estimate' throughout.

and Moreira 2006:1), making the affiliation assumption, as Milgrom and Weber point out, necessarily restrictive. Although, as they say, it may accord well with the qualitative features of some situations, such as the sale of works of art, there are many other situations where it does not (Monteiro and Moreira 2006:1; de Castro 2004). In fact, de Castro (2004) is particularly critical, observing that the affiliated values assumption is "very restrictive"; much more cumbersome to manipulate theoretically, with the monoticity of equilibrium hard to maintain; and leading to conclusions that are misleading if we try to apply them to reality. In his view, a return to the search for non-monotonic equilibria is urgently needed, citing Araujo, de Castro and Moriera's (2003) general existence result of non-monotonic symmetric equilibria with independent types. Araujo, de Castro and Moreira (2004), among others, have continued this work to examine multidimensional situations. Meanwhile Lebrun (1996, 1999) has obtained some results for asymmetric first price auctions, that is when bidders' item values are differently distributed, while Cantillon (2004) has considered both first and second price asymmetries. Guth *et al* (2005) provide a summary of much of the asymmetry work. No treatment appears yet to have been made of the equilibria for an asymmetric general independent values model where bidders have both different item values *and* imperfect associated estimates – most likely because of the difficulties involved in finding analytical solutions (Rothkopf *et al* 2003: 72).

Unlike AT, the goal of the DT approach is to maximise profits by the more practical means of mark-up manipulation in what is often termed "cost-plus pricing"[5]. This involves finding equilibria in DT-like scalar strategies rather than AT functions (Rothkopf *et al* 2003:73). Equilibrium multiplicative mark-up strategies in a *symmetric* CV sealed bid game theoretic setting have been reported in several studies. Rothkopf (1969, 1980a), for example,

[5] Hanson (1992), for example, mentions that "researchers report a sizable number of companies that use cost-plus pricing". There is also a considerable literature recording and advocating the use of a cost plus a percentage or dollar markup or combination of the two in practice. Eichner (1973), for example refers to "the overwhelming empirical evidence that most large business firms set their prices on the basis of a certain percentage mark-up above costs".

solves the n bidder Weibull distributed first price auction (FPA) situation analytically, while Oren and Rothkopf (1975), extend this to the situation where a bidder's strategy in one auction affects his competitors' behaviour in subsequent auctions, modelling bidding in a sequence of auctions as a multistage control process. Smith and Case (1975), on the other hand, consider the two bidder loglogistic CV FPA situation for both pure and mixed (randomised) strategies, while Rothkopf (1991) also considers the n bidder CV Weibull FPA and second price auction (SPA) situations in which bidders may submit two or more bids and then withdraw some bids after bids are opened.

No equilibrium results have been reported for scalar strategies other than multiplicative, with the exception of Rothkopf (1980b), who found, analytically, the equilibrium linear (affine) FPA mark-ups in the Weibull CV n bidder situation. In general, however, it is concluded that, despite the difficulties involved in equilibrium modelling with value uncertainties, multiplicative mark-up models at least have had some success. In this paper, we follow Rothkopf (1980b) in considering both the equilibrium multiplicative *and* additive mark-ups within a general linear mark-up strategy.

To do this, the starting point was to return to the DT original theme and consider the model where bidders have, independently, both different item values and their imperfect estimates. This is developed in this paper in the form of a generalised bidding model for *highbid* and *lowbid* auctions to include the major AT equilibrium models and DT models of bidding. From an AT perspective, this involves three major departures from the standard underlying assumptions:

(1) Restriction of the bidding function to a linear (affine) mark-up. DT assumes (in line with bidding practice) that bids are generated by applying a mark-up, in the form of an addition and/or multiplication, to the estimated item value. Standard AT, however, rarely makes such specific assumptions. A formless bid function is normally assumed, with an

explicit mark-up function seldom being prescribed. This suggests some modification is needed to standard AT to better reflect common practice. As Satterthwaite *et al* (2012) put it - "A procedure that endures in practice and seems to perform well but not [necessarily] optimally in a theoretical sense compels a reappraisal of the optimization analysis". Recent AT work by Compte and Postlewaite has pursued this theme. For example, they examine the effects of "strategy restrictions", where bidders are less concerned with maximising than following a set of rules they have learned or been taught the circumstances under which each of the rules is optimal (Compte and Postlewaite 2012). With the exceptions noted above by Rothkopf and others, however, equilibrium mark-up solutions are not yet known[6].

(2) Fixed difference independent values. The general DT literature, starting with Friedman (1956), assumes the difference between the expected value of each bidder's true costs (item values) to be fixed and non-zero – in contrast with IPV (where differences are assumed to be random) and CV (where differences are assumed to be zero)

(3) The km^{th} price award. A common assumption is an auction mechanism where the best ranked bid wins the item. However, Myerson (1981), for example, has shown the limitations of the using this as an allocation rule. Several alternatives have been reported in practice too, including "eliminating the highest bidder and awarding the contract with a 'second-highest-bid-wins' rule" (Switzerland)), eliminating the two highest bids (Italy) and the use of median bid auctions (Taiwan) (Decarolis 2009). Hoppe & Moldovanu (2009) have investigated this analytically in terms of m^{th} highest and median-bid-wins auctions in a general function IPV setting. k^{th} price auctions, on the other hand, where the payment for the winning bid is the value of the k^{th} ranked bid, have also received analytical treatment (e.g. Wolfstetter, 1996, 2001). Taken together, these suggest the

[6] Also worthy of note is that a linear strategy is often the best response to a linear strategy even under full rationality (e.g. Chatterjee and Samuelson, 1983).

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

need for a more general *km*th price auction, where the item is obtained by the *m*th ranked bidder at the amount of the *k*th ranked bid.

From a DT perspective, the most important additional consideration is the need to provide for the dynamics of the game playing situation – that each bidder needs to allow for the potential actions of opponents, who also allow for the potential actions of the bidder, etc *ad infinitum*.

This paper is organised as follows. First a generalised bidding model is developed to calculate the bidder's expected profit and auctioner's expected revenue/payment for both Independent Private Value (IPV) and General Independent Value (GIV) *km*th price sealed bid auctions (where the *m*th bidder wins at the *k*th bid payment) using a linear mark-up function. The Common Value (CV) assumption, and *highbid* and *lowbid* symmetric and asymmetric First Price Auctions (FPA) and Second Price Auctions (SPA) are included as special cases. The *n* bidder symmetric analytical results are then provided for the *highbid* and *lowbid* uniform FPA IPV and GIV models in equilibrium. Final comments concern implications, the assumptions involved and prospects for further research.

MODEL

Define a *km*th price auction as a sealed bid where the *m*th ranked bidder wins the item and the payment is the *k*th ranked bid. For the *highbid* auction, therefore, the *m*th highest bidder wins the item and the payment is the *k*th highest bid while, for the *lowbid* auction, the *m*th lowest bidder wins the item and the payment is the *k*th lowest bid. Let bidders be indexed $i = 1, \ldots, n$ $(n \geq 2)$ and bids $B_i = v_{1i} + S_i v_{2i}$ where S_i is an independent random variable denoting the unbiased estimated item value and v_{1i} and v_{2i} are (affine) mark-up manipulators. The *expected payment* is

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

$$\sum_{i=1}^{n} E\left[\left(t_{(k)} \middle| b_i = b_{(m)}\right)\right] \Pr\left(b_i = b_{(m)}\right) \tag{1}$$

and *expected profit* for bidder *i*

$$E_{(k,m)i} = \alpha\left(c_i - t_{(k)} \middle| b_i = b_{(m)}\right) \Pr\left(b_i = b_{(m)}\right) \tag{2}$$

where

- c_i is the bidder's actual item value
- $t_{(k)}$ is the payment contingent on winning
- $b_i = b_{(m)}$ indicates that bidder *i* has the m^{th} bid and wins the auction
- $\alpha = 1$ indicates a *highbid* auction and $\alpha = -1$ indicates a *lowbid* auction

If b_i is a value from a unique density function, $f_i(x)$, then

$$\Pr\left(b_i = b_{(m)}\right) = \int_{-\infty}^{\infty} g_i(x; m, \alpha) dx \tag{3}$$

$$E\left[\left(t_{(k)} \middle| b_i = b_{(m)}\right)\right] = \int_{-\infty}^{\infty} x h_i(x; k', m', \alpha) dx \Big/ \Pr\left(b_i = b_{(m)}\right) \qquad (k \neq m) \tag{4}$$

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

$$E\left[\left(t_{(m)} \middle| b_i = b_{(m)}\right)\right] = \int_{-\infty}^{\infty} x g_i(x; m, \alpha) dx \Big/ \Pr\left(b_i = b_{(m)}\right) \qquad (k = m) \tag{5}$$

$$c_{(DT)i} = E(S_i) \tag{6}$$

$$c_{(IPV)i} = \frac{E\left[\left(t_{(m)} \middle| b_i = b_{(m)}\right)\right] - v_{1i}}{v_{2i}} \tag{7}$$

where

$$g_i(x; m, \alpha) = f_i(x) \sum_{\substack{l_1=1 \\ l_1 \neq i}}^{n} \sum_{\substack{l_2=l_1+1 \\ l_2 \neq i}}^{n} \dots \sum_{\substack{l_{m-1}=l_{m-2}+1 \\ l_{m-1} \neq i}}^{n} \left[1 - \Phi_{l_1}(x)\right]\left[1 - \Phi_{l_2}(x)\right] \dots \left[1 - \Phi_{l_{m-1}}(x)\right] \prod_{\substack{o \neq i \\ o \neq l_1, l_2, \dots, l_{m-1}}}^{n} \Phi_o(x) \tag{8}$$

$$h_i(x; k, m, \alpha) = \sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(x) \sum_{\substack{l_1=1 \\ l_1 \neq i \\ l_1 \neq j}}^{n} \sum_{\substack{l_2=l_1+1 \\ l_2 \neq i \\ l_2 \neq j}}^{n} \cdots \sum_{\substack{l_{k-1}=l_{k-2}+1 \\ l_{k-1} \neq i \\ l_{k-1} \neq j}}^{n} \sum_{\substack{l_k=1 \\ l_k \neq i \\ l_k \neq j \\ l_k \neq l_1, l_2 \dots, l_{k-1}}}^{n} \sum_{\substack{l_{k+1}=l_{n-k+1}+1 \\ l_{k+1} \neq i \\ l_{k+1} \neq j \\ l_{k+1} \neq l_1, l_2, \dots, l_{k-1}}}^{n} \cdots \sum_{\substack{l_{m-2}=l_{m-3}+1 \\ l_{m-2} \neq i \\ l_{m-2} \neq j \\ l_{m-2} \neq l_1, l_2, \dots, l_{k-1}}}^{n}$$

$$\left[1 - \Omega_{l_1}(x)\right]\left[1 - \Omega_{l_2}(x)\right] \dots \left[1 - \Omega_{l_{k-1}}(x)\right] \int_{\psi}^{\phi} f_i(y) \left[\Omega_{l_k}(x) - \Omega_{l_k}(y)\right]\left[\Omega_{l_{k+1}}(x) - \Omega_{l_{k+1}}(y)\right] \dots \left[\Omega_{l_{m-2}}(x) - \Omega_{l_{m-2}}(y)\right] \prod_{\substack{o=1 \\ o \neq i \\ o \neq j \\ o \neq l_1, l_2, \dots, l_{m-2}}}^{n} \Omega_o(y) dy \tag{9}$$

$k' = \chi k + (n+1)(1-\chi)/2$ and $m' = \chi m + (n+1)(1-\chi)/2$

$\Phi(x) = (1-\alpha)/2 + \alpha F(x)$

$\Omega(x) = (1-\beta)/2 + \beta F(x)$

$\psi = (x-\infty)/2 - \beta(x+\infty)/2$

$\phi = (x+\infty)/2 + \beta(x-\infty)/2$

and

$\chi = 1$ if $m > k$

$\chi = -1$ if $k > m$

$\beta = 1$ if $\alpha m + (n+1)(1-\alpha)/2 > \alpha k + (n+1)(1-\alpha)/2$

$\beta = -1$ if $\alpha k + (n+1)(1-\alpha)/2 > \alpha m + (n+1)(1-\alpha)/2$

with the distribution function $F_i(x) = \int_{-\infty}^{x} f_i(u)du$

Substituting into (2) then gives

$$E_{(k=m)i} = \alpha \int_{-\infty}^{\infty} (c_i - x) g_i(x; m, \alpha) dx \tag{10}$$

$$E_{(k \neq m)i} = \alpha \left[\int_{-\infty}^{\infty} c_i g_i(x; m, \alpha) dx - \int_{-\infty}^{\infty} x h_i(x; k', m', \alpha) dx \right] \tag{11}$$

where $c_i = E[S_i]$ (e.g. Rothkopf 1969: 364) for the GIV assumption and $c_i = \frac{x - v_{1i}}{v_{2i}}$ for the IPV assumption. The expected payment is

$$R_k = \sum_{i=1}^{n} \int_{-\infty}^{\infty} x g_i(x; k, \alpha) dx \tag{12}$$

Note that for the symmetrical case, (8) and (9) simplify to

$$g_i(x; m, \alpha) = \frac{(n-1)!}{(n-m)!(m-1)!} f(x)[1 - \Phi(x)]^{m-1} [\Phi(x)]^{n-m} \tag{13}$$

$$h_i(x; k, m, \alpha) = \frac{(n-1)(n-2)!}{(k-1)!(m-k-1)!(n-m)!} f(x)[1 - \Omega(x)]^{k-1} \int_{\psi}^{\phi} f(y)[\Omega(x) - \Omega(y)]^{m-k-1} \Omega(y)^{n-m} dy \tag{14}$$

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Two special cases

For the FPA ($k = m = 1$), for example, (10) and (11) become

$$E_{(1,1)i} = \alpha \int_{-\infty}^{\infty} (c_i - x) f_i(x) \prod_{\substack{o=1 \\ o \neq i}}^{n} \left(\frac{1-\alpha}{2} + \alpha F_o(x) \right) dx \tag{15}$$

with

$$R_1 = \sum_{i=1}^{n} \int_{-\infty}^{\infty} x f_i(x) \prod_{\substack{o=1 \\ o \neq i}}^{n} \left(\frac{1-\alpha}{2} + \alpha F_o(x) \right) dx \tag{16}$$

while the SPA ($k = 2, m = 1$) is

$$E_{(2,1)i} = \alpha \left[\int_{-\infty}^{\infty} c_i f_i(x) \prod_{\substack{o=1 \\ o \neq i}}^{n} \left(\frac{1-\alpha}{2} + \alpha F_o(x) \right) dx - \int_{-\infty}^{\infty} x \left(\frac{1+\alpha}{2} - \alpha F_i(x) \right) \sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(x) \prod_{\substack{o=1 \\ o \neq i \\ o \neq j}}^{n} \left(\frac{1-\alpha}{2} + \alpha F_o(x) \right) dx \right] \tag{17}$$

with

$$R_2 = \sum_{i=1}^{n} \int_{-\infty}^{\infty} x f_i(x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1+\alpha}{2} - \alpha F_j(x) \right) \prod_{\substack{o=1 \\ o \neq i \\ o \neq j}}^{n} \left(\frac{1-\alpha}{2} + \alpha F_o(x) \right) dx \tag{18}$$

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society,
available online: https://doi.org/10.1057/jors.2013.163

Optimal mark-ups

It is assumed that all the bidders know: the value of *n*, *k*, *m*, α , $E(S_i)$ and the distribution form and parameters of $f_i(x)$ $(i = 1,2,\ldots,n)$. It is also assumed that each bidder knows his own estimated item value, s_i , - no one else does - and no one would ever revise his or her valuation when those of opponents are disclosed (Wolfsetter 1996:371). When *k*=*m*, for example, the optimal mark-ups for v_{1i} and v_{2i}, are obtained by finding the values that maximise (10) for a bidder. This can be done numerically for each bidder by maximising (10) after assuming some appropriate composite density form and suitable starting values, or analytically, by differentiating (10) with respect to v_{1i} and v_{2i}, setting to zero and solving for v_{1i} and v_{2i}. The Bayesian-Nash equilibrium mark-ups can be attempted numerically in the same way but repeating the procedure in turn for each bidder until and if convergence occurs. To do this analytically, involves differentiating (10), setting to zero and solving for v_{1i} and v_{2i} simultaneously for all bidders. The next section provides the equilibrium results for the symmetric uniform distribution.

THE SYMMETRIC UNIFORM EQUILIBRIUM

Linear mark up pricing

Assume bids are symmetrically and uniformly distributed with mean μ and variance σ^2 and supports $a = \mu - \alpha\sigma\sqrt{3}$ and $b = \mu + \alpha\sigma\sqrt{3}$. Then, from (10-12), for the IPV situation

$$E_{(k,m)} = \alpha\left[\left(\frac{k(b-a)}{n(n+1)} - \frac{b}{n}\right)v_2 - \frac{v_1}{n} - m\frac{b-a}{n(n+1)} + \frac{b}{n}\right] \tag{19}$$

$$R_k = \left(b - \frac{k(b-a)}{n+1} \right) v_2 + v_1 \tag{20}$$

The symmetrical version of the GIV situation is the CV model, where

$$E_{(k,m)} = \alpha \left[\left(\frac{k(b-a)}{n(n+1)} - \frac{b}{n} \right) v_2 - \frac{v_1}{n} + \frac{a+b}{2n} \right] \tag{21}$$

$$R_k = \left(b - \frac{k(b-a)}{n+1} \right) v_2 + v_1 \tag{22}$$

Table 1[7] provides the derivatives of (10-11) with respect to v_{1i} and v_{2i} and, setting $v_1 = v_{1i} = v_{1j}$ and $v_2 = v_{2i} = v_{2j}$, together with the equilibrium results for v_1^* and v_2^*. As is indicated, no equilibrium solutions exist for the CV case.

Table 2 summarises the results for all combinations of *n*, *m* and *k* values for the IPV situation in terms of the distribution moments. Equilibrium exists only for $m = 1$ and $m = n$ with the results being mirrored, the $R^*_{(m=1)}$ and $R^*_{(m==n)}$ trending towards opposite supports while $E^*_{(m-1)}$ and $E^*_{(m=n)}$ trend towards zero from positive and negative values respectively. Therefore the $m = 1$ mechanism, somewhat surprisingly, provides the best result for both the auctioner and the bidders *irrespective of the value of k*.

[7] Note the definition $0^0 = 1$ is used throughout this paper.

Additive mark-up pricing

Here, the model is

$$t_i = v_i + s_i \tag{23}$$

Setting $v_{2i} = 1$, for the IPV situation

$$E_{(k,m)} = \alpha\left[\frac{(n-m+1)b+ma}{n(n+1)} - \frac{b+v_1}{n} + \frac{k(b-a)}{n(n+1)}\right] \tag{24}$$

$$R_k = v_1 + \frac{a+b}{2} + \frac{n-2k+1}{n+1}\cdot\frac{b-a}{2} \tag{25}$$

and, for the CV situation

$$E_{(k,m)} = \alpha\left[-\frac{n-2k+1}{n(n+1)}\cdot\frac{b-a}{2} - \frac{v_1}{n}\right] \tag{26}$$

$$R_k = v_1 + \frac{a+b}{2} + \frac{n-2k+1}{n+1}\cdot\frac{b-a}{2} \tag{27}$$

Table 3 provides the derivatives with respect to v_{1i} together with the equilibrium results. Table 4 summarises the results for all combinations of *n*, *m* and *k* values in terms of the distribution moments. In this case, equilibrium exists for $m = 1$ and $m = n$ for both the IPV and CV situations. For the IPV, $R^*_{(m=1)}$ and $R^*_{(m==n)}$ trend towards opposite supports while

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

$E^*_{(m-1)}$ and $E^*_{(m=n)}$ trend towards zero from positive and negative values respectively. Therefore the $m=1$mechanism again provides the best result for both the auctioneer and the bidders in the IPV situation. The trends are reversed for the CV the situation, with the $m=n$ mechanism providing the best result for the auctioner, but with the $m=1$mechanism still being the best for the bidder.

In contrast with linear mark-up pricing, the value of k has an influencing effect in this case, with $R^*_{(m=1)}$ being maximised at $k=n$ but with an associated minimisation of $E^*_{(m-1)}$ for the IPV situation. Similarly, for the CV situation, with $R^*_{(m=n)}$ is maximised at $n=m=k$ but with an associated minimisation of $E^*_{(m=n)}$.

Multiplicative mark-up pricing

This time, the model is

$$t_i = v_i s_i \tag{28}$$

Setting $v_{1i} = 0$, for the IPV situation

$$E_{(k,m)} = \alpha\left[-\frac{a+b}{2}\cdot\frac{v_2-1}{2}+\frac{(2k-n-1)v_2+n-2m+1}{n(n+1)}\cdot\frac{b-a}{2}\right] \tag{29}$$

$$R_k = \left(\frac{a+b}{2}+\frac{n-2k+1}{n+1}\cdot\frac{b-a}{2}\right)v_2 \tag{30}$$

and, for the CV situation

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

$$E_{(k,m)} = \alpha\left[-\frac{a+b}{2}\cdot\frac{v_2-1}{2}+\frac{(2k-n-1)v_2}{n(n+1)}\cdot\frac{b-a}{2}\right] \tag{31}$$

$$R_k = \left(\frac{a+b}{2}+\frac{n-2k+1}{n+1}\cdot\frac{b-a}{2}\right)v_2 \tag{32}$$

Table 5 provides the derivatives with respect to v_{2i} together with the equilibrium results. Unlike the linear and additive pricing mechanisms, however, there are no simple results in terms of distribution moments for multiplicative mark ups due to the additive form of the expected profit and payment adopted in (2). To continue this general approach, a multiplicative version of (2) is needed such as

$$E'_{(k,m)i} = \alpha\left(\frac{c_i}{t_{(k)}\big|b_i = b_{(m)}}\right)\Pr\left(b_i = b_{(m)}\right) \tag{33}$$

An alternative used by Maskin & Riley (2000b), for instance, is to consider the special case where one of the distribution supports is zero. That is, either *a*=0 or *b*=0. Setting *a*=0 provides the results, for the IPV situation

$$E_{(k,m)} = \alpha\frac{b}{2n}\left(1-\frac{(n-2k+1)v_2-n+2m-1}{n+1}-v_2\right) \tag{31}$$

$$R_k = \frac{bv_2}{2}\left(1+\frac{n-2k+1}{n+1}\right) \tag{32}$$

and, for the CV situation

$$E_{(k.)} = \alpha \frac{b}{2n}\left(1 - \frac{(n-2k+1)v_2}{n+1} - v_2\right) \tag{31}$$

$$R_k = \frac{bv_2}{2}\left(1 + \frac{n-2k+1}{n+1}\right) \tag{32}$$

Table 7 summarises the results for all combinations of *n*, *m* and *k* values in terms of the distribution moments for high bid and low bid auctions. In this case, equilibrium exists for all mechanisms, both in the IPV and CV situations. For the IPV, $R^*_{(m=1)}$ and $R^*_{(m=n)}$ trend towards opposite supports while $E^*_{(m-1)}$ and $E^*_{(m=n)}$ trend identically towards zero from negative values for both high and low bid auctions. However the IPV $R^*_{(n>m>1)}$ and $E^*_{(n>m>1)}$ both differ for the high and low bid auctions, but providing the best $R^*_{(n>m>1)}$ and $E^*_{(n>m>1)}$ providing the best result for the auctioner and bidder in both cases for *m*=*n*-1. For the CV situation, $R^*_{(n>m>1)}$ and $E^*_{(n>m>1)}$ provide the best result for the auctioner and bidder ($E^* = 0$) together with $R^*_{(m=n)}$, $E^*_{(m=n)}$, $R^*_{(m=1)}$ and $E^*_{(m-1)}$ for high and low bid auctions respectively.

DISCUSSION

Correspondence with previous work

Surprisingly, despite an overwhelming amount of previous work in AT, very few absolute quantitative results have been reported. One exception is Klemperer's *general bidding function* in the FPA *highbid* ($\alpha = 1$) situation, where the equilibrium bid corresponds with Table 2 as $\frac{a}{n} + \frac{n-1}{n}s = a + \frac{n-1}{n}(s-a)$ (where $a = \mu - \sigma\sqrt{3}$) (Klemperer 1999: 57).

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Another is Maskin and Riley (2000a), whose numerical analysis gives $R_1^* = 0.33$ for the uniform symmetrical IPV 2 bidder *highbid* auction with $\mu = 0.5$, $\sigma = \frac{1}{2\sqrt{3}}$, which again corresponds with Table 2. Skitmore (2008) has also provided some numerical results for FPA and SPA special cases for the uniform and normal symmetrical and asymmetrical densities. For example, the equilibrium additive mark=up for the symmetrical *lowbid* uniform IPV FPA is given as $v_1^* = \sigma\sqrt{3}$, $E^* = \frac{2\sigma\sqrt{3}}{n(n+1)}$ and $R^* = \mu + \frac{n}{n+1}E^*$, again corresponding with Table 4.

Best mechanism

As Tables 2, 4 and 6 show, the values of R^* and E^* are determined solely by λ and γ in the expressions

$$R_k^* = \mu + \lambda\alpha\sigma\sqrt{3}$$

and

$$E_{(k,m)}^* = \gamma\sigma\sqrt{3}$$

This enables the best mechanisms to be identified independently of μ and σ as the mechanism with the highest ranking λ value identifies the best R^* and hence the best mechanism for rational auctioneers. The mechanism with the highest ranking γ value on the other hand, identifies the best E^* and hence the best mechanism for rational bidders. Table 7 summarises these best mechanisms for both R^* and E^* for the three mark-up pricing methods in *highbid* and *lowbid* auctions in IPV and CV situations. Of note here is that E^* is positive for all the R^* best mechanisms identified except for *n*=2. Also, the best mechanism is independent of *n* for *n*>3.

Mark-up hopping

In attempting to apply these results in practice it is obvious that, although the auctioner invariably chooses the auction mechanism, only the bidders can choose the pricing method to use. Therefore, although the auctioner may choose the best mechanism from Table 7 on the assumption, say, that the bidders will use linear mark-up pricing, the bidders may instead opt to use a different pricing method in order to increase their E^* values. As a result, the auctioner now becomes a player. For example, for the IPV high bid *n*=4 auction, the auctioner's best mechanism is *k*=4, *m*=1, for which λ is maximum in equilibrium at 0.4 when bidders are additive mark-up pricing. However, for the bidders, the *n*=*k*=4, *m*=1 mechanism produces a γ of only 0.05 when additive mark-up pricing, in comparison with $\gamma = 1$ when multiplicative or linear mark-up pricing. If, in the knowledge of this, the bidders then switch to one of the other two pricing methods, the auctioner's λ then drops to 0.20. To counter this, the auctioner may then consider the second best *n*=4 γ value, which is $\gamma = 0.30$ at *n*=4, *k*=3, *m*=1, again assuming additive mark-up pricing. Yet again though, it profits the bidders more to use multiplicative or linear pricing as both these again improve γ from 0.05 to 1 and leaves the auctioner again with $\lambda = 0.20$. So finally the auctioner realises there are now no further options available that will yield $\lambda > 0.20$ and examines all the mechanisms that will produce $\lambda = 0.20$. These comprise all the *m*=1 mechanisms. However, for *n*=4, *k*=*m*=1, the bidders maximise their expected profit to $\gamma = 1.25$ additive mark-up pricing, the effect of which is to reduce the auctioner's λ even further to 0.10. The auctioner's best mechanism in this case is therefore $n = 4 \geq k > m = 1$, which produces $\lambda \geq 0.20$. Extending this argument to all other situations provides the general solution for the best mechanism for uniform IPV auctions of $n \geq k > m = 1$ for *n*>2. For *n*=2, a similar line of reasoning indicates the best mechanism to be $m = 2$, with $k \geq 1$.

"Truth telling strategies"

Of particular interest here is the extent to which Vickrey's "truth-telling" effect still prevails in the mark-up model presented in this paper. That is where $v_1^* = 0$ and $v_2^* = 1$. Inspection of Table 2 indicates this to indeed be the case for the IPV *m*=1, *k*=2 (SPA) mechanism for linear mark-up pricing, with the same result also occurring asymptotically as *n* becomes large. The same result also occurs for the IPV additive mark-up (Table 4) but with the *m*=1, *k*=1 (FPA) mechanism applying for the CV situation, while for the multiplicative mark-up, the SPA mechanism again prevails in both IPV *highbid* and *lowbid* situations, but not in either of the CV *highbid* or *lowbid* situations.

Behaviour of R^* and E^*

Some further comments concerning R^* and E^* are also worthy of mention. Firstly, in addition to the finding that the relative R^* and E^* values are independent of μ and σ that is, they can be expressed in terms of $R_k^* = \mu + \lambda\alpha\sigma\sqrt{3}$ and $E_{(k,m)}^* = \gamma\sigma\sqrt{3}$ (under the assumption that either $\mu \pm \sigma\sqrt{3} = 0$ for multiplicative mark-ups):

- R^* generally tends asymptotically to the distribution support as *n* increases. One exception of this occurs with the CV additive mark-up result, where $\lambda_{(1,1)} = \frac{2n}{n+1}$ (Table 7)
- Increases in R^* are commensurate with decreases in E^* (Table 7)
- Although the values of v_1^* differ for *highbid* and *lowbid* IPV multiplicative mark-ups, the R^* and associated E^* results are identical for the *m*=*n* and *m*=1 mechanisms, although this is not the case for the CV situation (Table 6).

Effect of increased variability/uncertainty

In contrast with previous findings (e.g., Flanagan and Norman, 1985), the results here clearly indicate that both R^* and E^* increase as σ increases – suggesting, counter intuitively, that *both auctioner and bidders benefit from increased variability/uncertainty*. This raises the possibility of auctioners and bidders not only exploiting auctions where the item value is less certain, but even deliberately distorting the amount of uncertainly involved. However, as noted below, this finding may be more a result of the assumptions made in the model than of any practical significance.

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

CONCLUDING REMARKS

This paper has presented a combined Independent Private Value (IPV) and General Independent Valuec (GIV) equilibrium linear mark-up model for *km*th price sealed bid auctions that encompasses the Common Values (CV) assumption, symmetric and asymmetric *highbid* and *lowbid* first and second price auctions, and additive and multiplicative mark-ups all as special cases. Analytical results for the equilibrium mark-up, expected profit and expected revenue/payment are provided for the uniform density for the *n* bidder symmetric IPV and GIV models. Work on the asymmetric situation will be reported in a later paper.

Although this work provides an advancement of AT towards the more practical situation existing in many applications, where a strategic mark-up is applied to an estimated value, it should be said that there is much to do yet before practical implementation[8]. Of particular concern is the amount of information assumed to be known. This includes the type and parameters of the probability distributions involved and their associated bidders for each auction. If these are to be estimated from past auction bids, what method of estimation is to be used? This raises the related and deeper question concerning the stability of the parameters involved. Bidders are known to change their behaviour over time, in responses to changes in economic, legal and social circumstances; changes in personnel, risk attitudes, goals and strategic approaches such as loss leaders aimed at increasing longer term market share, bidder interdependence, etc and yet models do not take such changes into account, or even distinguish between long and short term effects in general. Neither do models take into account the dynamics involved as the effects of bidding in one auction are fed into the decision making for the next auction. Likewise, current models do not take into account the costs of bidding and their compensation, opportunity costs, the effect of taking bribes, coalitions, etc.

[8] See Rothkopf and Harstad (1994) for a general survey.

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

A further major issue, identified by Thaler (1988), is what to do when some bidders do not behave in an optimal way. Will a bounded rationality approach be possible? What kind of treatment is needed for non-optimising liars and cheats or those who artificially manipulate bids to fool competitors into making incorrect assumptions? In the words of Rothkopf (1969: 370), "a spiteful or ignorant competitor can make [equilibrium mark-up strategies] useless". Finally, in view of what has been said above, to what extent is the equilibrium assumption itself justified? Given the dynamics involved, it may be more realistic to consider models aimed more at survival than profit maximisation.

REFERENCES

Araujo A, de Castro LI and Moreira H (2003). Nonmonotonic auctions, Working paper. Instituto Nacional de Mathematica Pura Aplicada.

Araujo A, de Castro LI and Moreira H (2004). Pure strategy equilibria of multidimensional non-monotonic auctions. Working Paper, IMPA.

Cantillon E (2004). The effect of bidders' asymmetries on expected revenue in auctions. *Games and Economic Behavior* **62**(1): 1-25.

Cassady R Jr (1967). *Auctions and auctioneering.* University of California Press, Berkeley and Los Angeles.

Compte O and Postlewaite A (2012). Simple auctions. mimeo: http://www.enpc.fr/ceras/compte/auctionssecondversion.pdf

de Castro LI (2004). Is affiliation a good assumption? Working Paper, IMPA.

Decarolis F (2009). When the highest bidder loses the auction: theory and evidence from public procurement. *Bank of* Italy Temi di Discussione (Working Paper) No, 717.

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Eichner AS (1973). A theory of the determination of the mark-up under oligopoly. *The Economic Journal* **83**(332): 1184-1200.

Flanagan R and Norman G (1985). Sealed bid auctions: an application to the building industry. *Construction Management and Economics* **3**: 145-161.

Friedman L (1956). A competitive bidding strategy. *Operations Research* **1**(4): 104-112.

Gates M (1967). Bidding strategies and probabilities. *ASCE Journal of the Construction Division* **93**(CO1) Mar: 75-107.

Griesmer J. Levitan R and Shubik M (1967). Toward a study of bidding processes, Part IV – Games with unknown costs. *Naval Research Logistics Quarterly* **14:** 415-434.

Guth W, Ivanova-Stenzel R and Wolfsetetter E (2005). Bidding behavior in asymmetric auctions: An experimental study. *European Economic Review* **49**(7): 1891-1913.

Hanson W (1992). The dynamics of cost-plus pricing. *Managerial and Decision Economics* **13**(2): 149-161.

Hoppe K and Moldovanu B (2009). Auctions with non-monotone prize structures. Bachelor thesis, University of Bonn, http://konrad-hoppe.com/BachelorThesisPublic.pdf

Klemperer PD (1999). Auction theory: a guide to the literature. *Journal of Economic Surveys* **13**(3): 227-286.

Lebrun B (1996). Existence of an equilibrium in first price auctions. *Economic Theory* 7: 421-443.

Lebrun B (1999). First price auctions in the asymmetric N bidder case. *International Economic Review* **40**(1): 125-142.

Maskin E and Riley JG (2000a). Equilibrium in sealed high bid auctions. *Review of Economic Studies* **67**(3): 439-454.

Maskin E and Riley JG (2000b). Asymmetric auctions. *Review of Economic Studies* **67**(3): 413-438.

Milgrom PR (1981). Rational expectations, information acquisition and competitive bidding. *Econometrica* **49:** 921-943.

Milgrom, PR (1979). A convergence theorem for competitive bidding with differential information. *Econometrica* **47**: 679-688.

Milgrom PR and Weber RJ (1982). A theory of auctions and competitive bidding. *Econometrica* **50**(5): 1089-1122.

Monteiro PK and Moreira H (2006). First price auctions without affiliation. *Economic Letters* **91**: 1-7.

Myerson RB (1981). Optimal auction design. *Mathematics of Operations Research* **6**: 58-73.

Oren SS and Rothkopf MH (1975). Optimal bidding in sequential auctions. *Operations Research* **23**: 1080-1090.

Riley JG and Samuelson WF (1981). Optimal auctions. *American Economic Review* **71**: 381-392.

Rothkopf, M.H. (1969), A model of rational competitive bidding. *Management Science* **15**: 362-373.

Rothkopf MH (1980a). On multiplicative bidding strategies, *Operations Research* **28**: 570-575.

Rothkopf MH (1980b). Equilibrium linear bidding strategies. *Operations Research* **2**: 576-583.

Rothkopf MH (1991). On auctions with withdrawable winning bids. *Marketing Science* **10**: 40-57.

Rothkopf MH and Harstad RM (1994). Modeling competitive bidding: a critical essay. *Management Science* **40**(3): 364-384.

Rothkopf MH, Harstad RM and Yuhong Fu (2003). Is subsidizing inefficient bidders actually costly? *Management Science* **49**: 71-84.

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Runeson G and Skitmore RM (1999). Tendering theory revisited. *Construction Management and Economics* **17**(3): 285-296.

Satterthwaite M, Williams SR and Zachariadis KE (2012). Optimality versus practicality in market design: a comparison of two double auctions. mimeo: https://netfiles.uiuc.edu/swillia3/www/research/2012-06-26optimalityvspracticality.pdf

Skitmore RM (2008). First and second price independent values sealed bid procurement auctions: some scalar equilibrium results. *Construction Management and Economics* **26**(8): 785-801.

Smith BT and Case JH (1975). Nash equilibria in sealed bid auctions. *Management Science* **22**: 487-497.

Thaler RH (1988). The winner's curse. *Journal of Economic Perspectives* **2**: 191–202.

Vickrey W (1961). Counterspeculation, auctions and competitive sealed tenders, *Journal of Finance.* **16**: 8-37.

Wilson R (1969). Competitive bidding with disparate information. *Management Science* **15**: 446-448.

Wilson R (1977). A bidding model of perfect competition. *Review of Economic Studies* **44**: 511-518.

Wolfstetter E (1996). Auctions: an introduction. *Journal of Economic Surveys* **10**(4): 367-420.

Wolfstetter E (2001). Third-and lower-price auctions. In *Beiträge zur Mikro-und zur Makroökonomik* (pp. 495-503). Springer Berlin Heidelberg.

Type	Model	Derivative wrt v_{1i} proportional to	Derivative wrt v_{2i} proportional to	v_1^*	v_2^*
$k = m$	IPV $\left(c_i = \frac{x - v_{1i}}{v_{2i}}\right)$	$\frac{a(v_2 - 1) + v_1}{(b-a)v_2} 0^{n-m} - \frac{b(v_2 - 1) + v_1}{(b-a)v_2} 0^{m-1} - \frac{1}{nv_2}$	$a \frac{a(v_2 - 1) + v_1}{(b-a)v_2} 0^{n-m} - b \frac{b(v_2 - 1) + v_1}{(b-a)v_2} 0^{m-1} + \frac{(n+1)[(v_2 - 2)b + v_1] - (b-a)(v_2 - 2)m}{n(n+1)v_2}$	$\frac{(n+1)b - (b-a)m - n(a0^{n-m} + b0^{m-1})}{n(0^{n-m} + 0^{m-1})}$	$\frac{n-1}{n} \frac{a^2 0^{n-m} + b^2 0^{m-1}}{(a0^{n-m} - b0^{m-1})^2 (0^{n-m} + 0^{m-1})}$
	CV $\left(c_i = \frac{a+b}{2}\right)$	$\frac{[2(av_2 + v_1) - a - b]0^{n-m} - [2(bv_2 + v_1) - a - b]0^{m-1}}{2(b-a)v_2}$	$\frac{a[2(av_2 + v_1) - a - b]0^{n-m} - b[2(bv_2 + v_1) - a - b]0^{m-1}}{2(b-a)v_2} + \frac{2(bv_2 + v_1) - a - b}{2nv_2} - \frac{m(b-a)}{n(n+1)}$	Indeterminate (v_1 can be any value for equilibrium)	Indeterminate (depends on v_1)
$m > k$	IPV $\left(c_i = \frac{x - v_{1i}}{v_{2i}}\right)$	$\left[\frac{bv_2 + v_1 - a}{(b-a)v_2} - \frac{k}{n}\right]0^{n-m} - \frac{1}{nv_2}$	$-a\left[\frac{k}{n} + \frac{a - bv_2 - v_1}{(b-a)v_2}\right]0^{n-m} + \frac{(n+1)[b(v_2 - 2) + v_1] - (b-a)(kv_2 - 2m)}{n(n+1)v_2}$	$-b\left(\frac{n-1}{k} - 1\right) iff\ m = n$	$\frac{n-1}{k} iff\ m = n$
	CV $\left(c_i = \frac{a+b}{2}\right)$	$\left(\frac{2(bv_2 + v_1) - a - b}{2(b-a)v_2} - \frac{k}{n}\right)0^{n-m}$	$a\left(\frac{2(bv_2 + v_1) - a - b}{2(b-a)v_2} - \frac{k}{n}\right)0^{n-m} + \frac{1}{n}\left(\frac{2(bv_2 + v_1) - a - b}{2v_2} - \frac{k(b-a)}{n+1}\right)$	No equilibrium exists (no turning point for any v_1 value)	Indeterminate (depends on v_1)
$k > m$	IPV $\left(c_i = \frac{x - v_{1i}}{v_{2i}}\right)$	$\left[\frac{k-1}{n} - \frac{b(v_2 - 1) + v_1}{(b-a)v_2}\right]0^{m-1} - \frac{1}{nv_2}$	$b\left[\frac{k-1}{n} - \frac{b(v_2 - 1) + v_1}{(b-a)v_2}\right]0^{m-1} + \frac{(n+1)[b(v_2 - 2) + v_1] - (b-a)(kv_2 - 2m)}{n(n+1)v_2}$	$-a \frac{k-2}{n-k+1} iff\ m = 1$	$\frac{n-1}{n-k+1} iff\ m = 1$
	CV $\left(c_i = \frac{a+b}{2}\right)$	$\left\{\frac{k}{n} - \frac{2nv_1 + 2[(n+1)b - a]v_2 - n(a+b)}{2n(b-a)v_2}\right\}0^{m-1}$	$b \frac{n(a+b) - 2[nb - (k-1)(b-a)]v_2 - 2nv_1}{2n(b-a)v_2} 0^{m-1} + \frac{(n+1)(2bv_2 - a - b + 2v_1) - 2k(b-a)v_2}{2n(n+1)v_2}$	No equilibrium exists (no turning point for any v_1 value)	Indeterminate (depends on v_1)

Table 1: Derivatives with respect to v_1 and v_2 and linear mark-up equilibria

Type	v_1^*	v_2^*	E^*	R^*
$m=1$	$-\left(\frac{k-2}{n-k+1}\right)\left(\mu-\alpha\sigma\sqrt{3}\right)$	$\frac{n-1}{n-k+1}$	$\frac{2}{n(n+1)}\cdot\sigma\sqrt{3}$	$\mu+\frac{n-3}{n+1}\cdot\alpha\sigma\sqrt{3}$
$m=n$	$-\left(\frac{n-1}{k}-1\right)\left(\mu+\alpha\sigma\sqrt{3}\right)$	$\frac{n-1}{k}$	$-\frac{2}{n(n+1)}\cdot\sigma\sqrt{3}$	$\mu-\frac{n-3}{n+1}\cdot\alpha\sigma\sqrt{3}$
$n>m>1$	Indeterminate/no equilibrium	Indeterminate/no equilibrium	NA	NA

Table 2: IPV linear mark up results in terms of moments

Type	Model	Derivative wrt v_{1i} proportional to	v_1^*
$k = m$	IPV $(c_i = x - v_{1i})$	$\frac{0^{n-m} - 0^{m-1}}{b-a} v_1 - \frac{1}{n}$	$\frac{b-a}{n(0^{n-m} - 0^{m-1})}$
	CV. $\left(c_i = \frac{a+b}{2}\right)$	$\left(\frac{v_1}{b-a} - \frac{1}{2}\right)0^{n-m} - \left(\frac{v_1}{b-a} + \frac{1}{2}\right)0^{m-1}$	$\frac{1}{2} \cdot \frac{(b-a)(0^{n-m} + 0^{m-1})}{0^{n-m} - 0^{m-1}}$
$m > k$	IPV $(c_i = x - v_{1i})$	$\left(1 - \frac{k}{n} + \frac{v_1}{b-a}\right)0^{n-m} - \frac{1}{n}$	$\frac{b-a}{n0^{n-m}} - \frac{(n-k)(b-a)}{n}$
	CV. $\left(c_i = \frac{a+b}{2}\right)$	$\left(\frac{1}{2} - \frac{k}{n} + \frac{v_1}{b-a}\right)0^{n-m}$	$-\frac{n-2k}{2n}(b-a) \quad iff \;\; n = m$
$k > m$	IPV $(c_i = x - v_{1i})$	$\left(\frac{k-1}{n} - \frac{v_1}{b-a}\right)0^{m-1} - \frac{1}{n}$	$\frac{b-a}{n}\left(k - 1 - \frac{1}{0^{m-1}}\right)$
	CV. $\left(c_i = \frac{a+b}{2}\right)$	$\left(\frac{k}{n} - \frac{n+2}{2n} - \frac{v_1}{b-a}\right)0^{m-1}$	$\frac{b-a}{n}\left(k - \frac{n}{2} - 1\right) \quad iff \;\; m = 1$

Table 3: Derivatives with respect to v_1 and additive mark-up equilibria

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Type	Model	v_1^*	E^*	R^*
$m=1$	IPV $(c_i = x - v_{1i})$	$\frac{2(k-2)}{n}\cdot\alpha\sigma\sqrt{3}$	$\frac{2(n-k+2)}{n^2(n+1)}\cdot\sigma\sqrt{3}$	$\mu+\left(\frac{n^2-3n+2k-4}{n(n+1)}\right)\alpha\sigma\sqrt{3}$
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	$-\frac{n-2k+2}{n}\cdot\alpha\sigma\sqrt{3}$	$\frac{2(n-k+1)}{n^2(n+1)}\cdot\sigma\sqrt{3}$	$\mu-\left(\frac{2(n-k+1)}{n(n+1)}\right)\alpha\sigma\sqrt{3}$
$m=n$	IPV $(c_i = x - v_{1i})$	$-\frac{2(n-k-1)}{n}\cdot\alpha\sigma\sqrt{3}$	$-\frac{2(n+k+1)}{n^2(n+1)}\cdot\sigma\sqrt{3}$	$\mu-\left(\frac{n^2-n-2k-2}{n(n+1)}\right)\alpha\sigma\sqrt{3}$
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	$-\frac{n-2k}{n}\cdot\alpha\sigma\sqrt{3}$	$-\frac{2k}{n^2(n+1)}\cdot\sigma\sqrt{3}$	$\mu+\left(\frac{2k}{n(n+1)}\right)\alpha\sigma\sqrt{3}$
$n>m>$	IPV $(c_i = x - v_{1i})$	No turning point	NA	NA
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	No turning point	NA	NA

Table 4: Additive mark-up results in terms of moments

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society, available online: https://doi.org/10.1057/jors.2013.163

Type	Model	Derivative wrt v_{2i} proportional to	v_2^*
$k=m$	IPV $\left(c_i=\frac{x}{v_{2i}}\right)$	$\frac{\left(a^2 0^{n-m}-b^2 0^{m-1}\right)(v_2-1)}{(b-a)v_2}+\frac{(ma-mb+b+nb)(v_2-2)}{n(n+1)v_2}$	$\frac{n(n+1)\left(a^2 0^{n-m}-b^2 0^{m-1}\right)+2(b-a)(ma-mb+b+nb)}{n(n+1)\left(a^2 0^{n-m}-b^2 0^{m-1}\right)+(b-a)(ma-mb+b+nb)}$
	CV $\left(c_i=\frac{a+b}{2}\right)$	$\frac{a(2av_2-a-b)0^{n-m}-b(2bv_2-a-b)0^{m-1}}{2(b-a)v_2}-m\frac{b-a}{n(n+1)}+\frac{2bv_2-a-b}{2nv_2}$	$\frac{1}{2}\cdot\frac{na(a+b)0^{n-m}-(n+1)b0^{m-1}+(n+1)(a+b)(b-a)}{n(n+1)\left(a^2 0^{n-m}-b^2 0^{m-1}\right)+(b-a)(ma-mb+b+nb)}$
$m>k$	IPV $\left(c_i=\frac{x}{v_{2i}}\right)$	$a\left[\left(\frac{b}{b-a}-\frac{k}{n}\right)-\frac{a^2}{(b-a)v_2}\right]0^{n-m}+\frac{(n+1)b(v_2-2)-(kv_2-2m)(b-a)}{n(n+1)v_2}$	$\frac{n(n+1)a^2 0^{n-m}+2(b-a)(ma-mb+b+nb)}{(n+1)a(ka-kb+nb)0^{n-m}+(b-a)(ka-kb+b+nb)}$
	CV $\left(c_i=\frac{a+b}{2}\right)$	$a\left[\left(\frac{b}{b-a}-\frac{k}{n}\right)-\frac{a+b}{2(b-a)v_2}\right]0^{n-m}+\frac{(n+1)(2bv_2-a-b)-2kv_2(b-a)}{2n(n+1)v_2}$	$\frac{1}{2}\cdot\frac{(n+1)(a+b)\left(na0^{n-m}-a+b\right)}{\left(a0^{n-m}(n+1)-a+b\right)(ka-kb+nb)+b(b-a)}$
$k>m$	IPV $\left(c_i=\frac{x}{v_{2i}}\right)$	$b\left[\frac{k-1}{n}-\frac{b(v_2-1)}{(b-a)v_2}\right]0^{m-1}+\frac{(n+1)b(v_2-2)-(kv_2-2m)(b-a)}{n(n+1)v_2}$	$\frac{n(n+1)b^2 0^{m-1}-2(b-a)(ma-mb+b+nb)}{(n+1)b(ka-kb+nb-a+b)0^{m-1}-(b-a)(ka-kb+b+nb)}$
	CV $\left(c_i=\frac{a+b}{2}\right)$	$b\left[\frac{k-1}{n}+\frac{a+b-2bv_2}{2(b-a)v_2}\right]0^{m-1}+\frac{(n+1)(2bv_2-a-b)-2kv_2(b-a)}{2n(n+1)v_2}$	$\frac{1}{2}\cdot\frac{(n+1)(a+b)\left(nb0^{m-1}+a-b\right)}{(n+1)b(ka-kb+nb-a+b)0^{m-1}-(b-a)(ka-kb+b+nb)}$

Table 5: Derivatives with respect to v_{2i} *and multiplicative mark-up equilibria*

Type	Model	v_2^* **highbid**	v_2^* **lowbid**	E^* **highbid**	E^* **lowbid**	R^* **highbid**	R^* **lowbid**
$m=1$	IPV $(c_i = x - v_{1i})$	$\frac{n-1}{n-k+1}$	$\frac{2}{k}$	$\frac{2}{n(n+1)}\sigma\sqrt{3}$	$\frac{2}{n(n+1)}\sigma\sqrt{3}$	$\mu + \frac{n-3}{n+1}\alpha\sigma\sqrt{3}$	$\mu + \frac{n-3}{n+1}\alpha\sigma\sqrt{3}$
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	$\frac{(n-1)(n+1)}{2n(n-k+1)}$	$\frac{n+1}{2k}$	$\frac{1}{n^2}\sigma\sqrt{3}$	0	$\mu - \frac{1}{n}\sigma\sqrt{3}$	μ
$m=n$	IPV $(c_i = x - v_{1i})$	$\frac{2}{n-k+1}$	$\frac{n-1}{n}$	$-\frac{2}{n(n+1)}\sigma\sqrt{3}$	$-\frac{2}{n(n+1)}\sigma\sqrt{3}$	$\mu - \frac{n-3}{n+1}\alpha\sigma\sqrt{3}$	$\mu - \frac{n-3}{n+1}\alpha\sigma\sqrt{3}$
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	$\frac{1}{2}\cdot\frac{n+1}{n-k+1}$	$\frac{(n-1)(n+1)}{2nk}$	0	$-\frac{1}{n^2}\sigma\sqrt{3}$	μ	$\mu + \frac{1}{n}\alpha\sigma\sqrt{3}$
$n > m > 1$	IPV $(c_i = x - v_{1i})$	$2\frac{n-m+1}{n-k+1}$	$\frac{2m}{k}$	$-2\frac{n-m+1}{n(n+1)}\sigma\sqrt{3}$	$\frac{2m}{n(n+1)}\sigma\sqrt{3}$	$\mu + \frac{3n-4m+3}{n+1}\alpha\sigma\sqrt{3}$	$\mu + \frac{n-4m+1}{n+1}\alpha\sigma\sqrt{3}$
	GIV (CV) $\left(c_i = \frac{a+b}{2}\right)$	$\frac{1}{2}\cdot\frac{n+1}{n-k+1}$	$\frac{n+1}{2k}$	0	0	μ	μ

Table 6: Multiplicative mark-up results in terms of moments

This is an Accepted Manuscript of an article published in 2014 by Taylor & Francis in Journal of the Operational Research Society,
available online: https://doi.org/10.1057/jors.2013.163

			λ **(R)**			γ **(E)**		
			n=2	***n=3***	***n>3***	***n=2***	***n=3***	***n>3***
IPV	Linear mark-up	*k*	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$
		m	*m=n*	$m \neq 2$	*m=1*	*m=1*	*m=1*	*m=1*
		equil	$\frac{1}{3}$	0	$\frac{n-3}{n+1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{n(n+1)}$
	Additive mark-up	*k*	*k=n*	*k=n*	*k=n*	*k=1*	*k=1*	*k=1*
		m	*m=n*	$m \neq 2$	*m=1*	*m=1*	*m=1*	*m=1*
		equil	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{n(n-1)-4}{n(n+1)}$	$\frac{1}{2}$	$\frac{2}{9}$	$\frac{2}{n^2}$
	Multiplicative (*highbid*)	*k*	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$
		m	*m=n*	*m=2*	*m=2*	*m=1*	*m=1*	*m=1*
		equil	$\frac{1}{3}$	1	$\frac{3n-5}{n+1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{n(n+1)}$
	Multiplicative (*lowbid*)	*k*	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$
		m	*m=n*	$m \neq 2$	*m=1*	*m=1*	*m=(n-1)*	*m=(n-1)*
		equil	$\frac{1}{3}$	0	$\frac{n-3}{n+1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2m}{n(n+1)}$
CV	Linear mark-up	*k*	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*
		m	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*
		equil	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*
	Additive mark-up	*k*	*k=n*	*k=n*	*k=n*	*k=1*	*k=1*	*k=1*
		m	*m=n*	*m=n*	*m=1*	*m=1*	*m=1*	*m=1*
		equil	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2n}{n+1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{n(n+1)}$
	Multiplicative (*highbid*)	*k*	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$
		m	*m=1*	*m=1*	*m=1*	*m=1*	*m=1*	*m=1*
		equil	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{n}$	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{n^2}$
	Multiplicative (*lowbid*)	*k*	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$	$n \geq k \geq 1$
		m	*m=n*	*m=n*	*m=n*	*m=1*	$m \neq n$	$m \neq n$
		equil	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{n}$	0	0	0

Table 7: Best mechanisms