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Published in:
Journal of Economics and Business

DOI:
10.1016/j.jeconbus.2016.01.003

Enhancing mean-variance portfolio selection by modeling distributional asymmetries☆

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Abstract

Why do mean-variance (MV) models perform so poorly? In searching for an answer to this question, we estimate expected returns by sampling from a multivariate probability model that explicitly incorporates distributional asymmetries. Specifically, our empirical analysis shows that an application of copulas using marginal models that incorporate dynamic features such as autoregression, volatility clustering, and skewness to reduce estimation error in comparison to historical sampling windows. Using these copula-based models, we find that several MV-based rules exhibit statistically significant and superior performance improvements even after accounting for transaction costs. However, we find that outperforming the naïve equally-weighted ($1/N$) strategy after accounting for transactions costs still remains an elusive task.

Keywords: mean-variance, portfolio management, copula, asymmetric marginals
JEL classification: G11, C16

☆The authors are extremely grateful for the advice and feedback received on earlier versions of this manuscript from: Andrew Ang, Barbara Ostdiek, Krishna Paudyal, and Tom Smith; as well as anonymous participants at the following presentations: Inaugural Universitas 21 Doctoral Research Conference in Business, University of Connecticut; Windham Capital Management; Leeds University Business School Research Seminar Series, University of Leeds; Birmingham Business School Doctoral Conference, University of Birmingham; Department of Mathematics Research Seminar Series, University of Oslo; Adam Smith School of Economics and Finance Research Seminar Series, University of Glasgow; 5th International Accounting & Finance Doctoral Symposium, University of Strathclyde; Scottish Doctoral Colloquium in Accounting and Finance, University of Strathclyde and the University of Queensland Business School Annual Research Colloquium. We also acknowledge the support of the high performance computing team within the Research Computing Centre at the University of Queensland and the Australian Research Council (DP0770081).

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1. Introduction

Mean-variance (MV) optimization (Markowitz, 1952), either assumes that portfolio returns are normally distributed or that investors exhibit quadratic utility preferences. As such, the theory is unable to account for the presence of higher moments beyond the mean and variance in both the portfolio returns distributions or investor preferences (Cremers et al., 2005). Thus, MV optimization is often criticized for having little practical use as it maximizes estimation error, produces unintuitive portfolio distributions, and extreme portfolio weights (Michaud, 1989). More recently, the empirical performance of MV optimization has been subject to intense scrutiny due to the findings of DeMiguel et al. (2009b) who show that the naïve equally-weighted ($1/N$) portfolio is able to outperform several advanced MV models over the long-term, in out-of-sample analyses across a broad range of data sets. But, can MV models perform better?

Our strategy for answering this question is to focus on the idea that optimal portfolio diversification is dependent upon the quality of the sample inputs into the MV model. Of particular interest are the asymmetries within the joint distribution of stock returns widely reported in the financial literature. These asymmetries manifest in the form of asymmetric volatility clustering (Glosten et al., 1993), skewness within the distribution of individual stock returns (Aït-Sahalia and Brandt, 2001) or as asymmetric dependence (Longin and Solnik, 2001; Ang and Chen, 2002; Patton, 2004). Asymmetric dependence describes the scenario in which asset returns exhibit stronger correlations during market downturns than during market upturns. Practitioners also describe this effect as asymmetric correlations and are concerned about it because it reduces the benefits of diversification when they are needed the most (Chua et al., 2009).

Our paper makes three key contributions to the literature. First, we document evidence that MV optimization is improved in relation to use of historical samples by managing asymmetries within the marginals and reducing estimation errors in the variance-covariance (VCV) matrix. Second, we are the first paper to apply copulas to several sophisticated extensions of the MV optimization rule that allows the identification of models that might be robust to higher moment risk. Third, by including the combination portfolio rules of Tu and Zhou (2011), we assess how beneficial the application of model-based estimates are for an applied finance investigation in portfolio management.

Empirical studies typically use historical sampling returns or simulations that, to their detriment, do not explicitly account for such asymmetries within the returns distribution when testing MV optimization models (Tu and Zhou, 2011; DeMiguel et al., 2009b; MacKinlay and Pastor, 2000). An inferior choice of the assumed data-generating process for samples used in the MV optimization process can lead to poor performance. Therefore, in this article, we simply ask: can we achieve performance improvements in MV optimization by enhancing the sample input models to capture asymmetries in the marginal distributions of returns? There are some encouraging signs from the recent literature in this regard. Thorp and Milunovich (2007) use predictions from asymmetric VCV forecasting models to calculate optimal weights for international equity portfolios. They find that investors who exhibit moderate levels of risk-aversion with longer re-balancing horizons benefit from using asymmetric forecasts. Their study is limited towards constructing three-asset MV portfolios comprising of two equity market returns (e.g., US, Japan, UK, and Australia) and the risk-free asset. DeMiguel et al. (2013b) finds that using option-implied volatility and skewness to adjust expected returns leads to an improvement in the Sharpe Ratio for MV optimization. Indeed, Markowitz (1952) explicitly recommends the use of a probability model to generate the inputs required by the MV model.

Portfolios generated by MV optimization use a sample VCV matrix as the Maximum Likelihood Estimator (MLE) due to the assumption of normally distributed returns. However, if the data deviates (even slightly) from normality, MLEs (e.g., VCV matrix) that are based on normality assumptions are not necessarily the most efficient (Huber and Ronchetti, 2009, Example 1.1). Fantazzini (2009) models returns data that exhibit asymmetries such as skewness with an elliptical copula (e.g., Gaussian and Student t) with intentionally misspecified symmetric marginals. He finds that the misspecification of the marginals can lead to severe negative biases (as much as 70% of the true values) in the correlation estimates when positive correlations are considered.[1] Such issues regarding efficiency and negative bias are of critical importance in portfolio selection where extensive evidence shows that the empirical distribution of returns usually deviates from normality (DeMiguel and Nogales, 2009).

Using historical returns samples to calculate the expected return and the VCV matrix increases the likelihood of estimation error. Therefore, we seek to understand if sampling from a joint distribution via a copula that links asymmetric marginals is able to reduce estimation error and negative bias in correlation estimates of returns for MV optimization in an applied finance study. Such an investigation complements the tests and analyses of Fantazzini (2009).

Specifically, our work applies the Gaussian copula as a parsimonious model for MV optimization that is scalable for portfolios of higher dimensions.[2] Asymmetries in the marginals are modeled using the GARCH-GJR (Glosten et al., 1993) model that is able to capture the leverage effect, namely, the tendency for volatility to increase more with negative news rather than with positive news. Skewness and kurtosis of the residuals are modeled using the Hansen (1994) Skewed Student t (Skew-T). Based on a combination of these models, we generate asset returns using Monte-Carlo simulations, which in turn are fed into the group of 15 MV optimization rules (investigated and developed by DeMiguel et al. (2009b) and Tu and Zhou (2011)).[3]

[1] As compared to positive correlations, Fantazzini (2009) find that the bias almost doubles for negative correlations.

[2] While asymmetric copulas exist and have been applied in portfolio management (Low et al., 2013) applications, they are really only useful in optimizations of utility functions where investors have higher moment preferences. The application of asymmetric copulas to MV optimization is more complex than is necessary for the current study since the inputs to MV optimization only involve the expected return and the VCV matrix, and do not involve any information from the tails of the returns distribution.

[3] The long-run analysis performed over multiple periods in our study is greatly aided by our home institution's distributed network of high performance parallel computing systems.

Our results show empirical evidence in support of incorporating out-of-sample marginal distribution asymmetries in improving the estimates of expected returns for MV optimization. Generally, we find that the incorporation of distributional asymmetries in returns estimates produce statistically superior performance outcomes beyond using historical samples for several MV strategies even after accounting for realistic transaction costs. Among these strategies, variants of the minimum-variance (MIN) and combination portfolio rules (Kan and Zhou, 2007; Tu and Zhou, 2011) exhibit the best performance. As the process of sample-based portfolio optimization generally requires substantial turnover in its implementation, after accounting for the impact of transaction costs, the application of our model-based estimates produces statistically different and improved outcomes compared to the $1/N$ strategy only for the data set involving individual stocks. Although these improvements come at the cost of increased turnover, the MV rules are easier to implement in practice as we find that the target portfolio weights exhibit reduced average standard deviation. In addition, our results are robust across sampling windows of different sizes.

The remainder of this study is structured as follows. Section 2 describes the US industry, international country, and individual US stock data sets that we employ. Section 3, describes our multivariate probability model that links non-elliptical marginals with an elliptical copula, and the MV optimization model, and list of MV model variants investigated. Section 4 presents the results of our empirical study and we conclude in Section 5.

2. Data

The data sets analyzed in this study are listed in Table 1. The international country and US industry data sets have previously been used by DeMiguel et al. (2009b) and Tu and Zhou (2011). However, we extend the time period investigated to include the highly volatile bear market of 2007-2009. As individual stocks have distributional properties that are different from international indices and industry portfolios, we also include a data set of the constituent stocks of the Dow 30 that form the US Dow Jones Industrial Average (DJIA). Notably, professional investment managers whose objective is to track the US S&P500 often do so via allocation amongst the 30 DJIA stocks. Similarly to DeMiguel et al. (2009b), we apply rolling sampling windows of 120 months. Our analysis includes a 240 month sampling window as Tu and Zhou (2011) find that longer sampling windows result in improved portfolio strategy performance.

The international country index data set consists of the US, Canada, Japan, France, Italy, Germany, Switzerland, UK and Australia. Returns are calculated based on the month-end US-dollar value of the country equity index, beginning January 1970 through to July 2010. The 17 US industry portfolio data set comprises Food, Mines, Oil, Clothing, Durables, Chemicals, Consumables, Construction, Steel, Fabricated Products, Machinery, Cars, Transportation, Utilities, Retail, Finance, and Other. The US DJIA stock data set is constructed similar to Preis et al. (2012) where the portfolio constituents are continuously updated over the sample period according to the changes to the DJIA as indicated by the records from the official Dow Jones website.[4] The chosen sample period for both the industry and the DJIA data begins July 1953 through to December 2010. Monthly returns for all data sets are calculated in excess of the risk-free rate as represented by the 90-day T-bill as listed on Ken French's website.[5,6]

Table 1: Data sets considered
This table shows the list of data sources where N denotes the total number of risky assets within the portfolio data set and the number after the "+" indicates the number of factor portfolios available. MKT is the market, SMB is the size-based mimicking portfolio, HML is the value/growth-based mimicking portfolio and MOM is the momentum-based mimicking portfolio.

Name	Source	N	Time period	Factors
International country indices	MSCI	9+1	01/1970-07/2010	World
US industrial indices	Ken French's website	17+4	07/1953-12/2010	MKT/SMB/HML/MOM
US DJIA stocks	CRSP	30+4	07/1953-12/2010	MKT/SMB/HML/MOM

The factor portfolio returns World, MKT (market), SMB (size-based mimicking portfolio), HML (value/growth-based mimicking portfolio), and MOM (momentum-based mimicking portfolio) are used to implement the Bayesian Data-and-Model (DM)[7] approach on several of our data sets. Specifically, we consider the international capital asset pricing model (ICAPM) with the international country indices and with the US industry portfolios, as well as the domestic CAPM, 3-factor (Fama and French, 1993) and 4-factor (Carhart, 1997) approaches. We do not include these factor returns as investable assets within our study.[8]

The international country and US industry portfolios are readily investable as index futures that are highly liquid financial instruments. The US DJIA portfolio consists of highly liquid, large-cap stocks that are heavily traded on the NYSE and NASDAQ. As a result, the portfolios investigated exhibit several attributes that are desirable for portfolio re-balancing by institutional investors such as minimal short-sales constraints, low transaction costs, and low-adverse selection costs (Balduzzi and Lynch, 1999).

In addition, our selection of data sets are well diversified with a broad coverage of assets across the entire US market or among the largest developed economies in the world. Theoretically, these portfolios are well-diversified and are most likely to approximate multivariate normality. However, Longin and Solnik (2001) show that international equities exhibit stronger positive correlations

[4]See "http://www.djindexes.com/".

[5]See "http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/".

[6]We consider the asset allocation exercise across risky assets only. The appendix of DeMiguel et al. (2009b) show that including investment in a risk-free asset does not lead to any optimization rule consistently outperforming the $1/N$ strategy. Furthermore, inclusion of the risk-free asset would imply that the performance of the overall portfolio would also depend on market-timing ability. This is contrary to our goal of designing an investment portfolio to perform persistently regardless of market conditions or regime.

[7]We describe the alternative MV rules investigated in Subsection 3.2.

[8]In their appendix, DeMiguel et al. (2009b) indicate that exclusion of these factors as investable assets does not impact the overall ranking of the $1/N$ strategy.

when the US market is going down than when it is going up, and Ang and Chen (2002) find strong asymmetric correlations between stock portfolios and the US market. Preis et al. (2012) find that the average correlation across stocks constituting the DJIA increases during times of market stress and high volatility. Thus, these studies indicate that the diversification effect that should protect a portfolio during periods of market turbulence is reduced when they are needed the most.

3. Research Method

Modern portfolio theory (Markowitz, 1952) states that portfolio management is a 2-stage process whereby portfolio managers first, produce estimates of the expected return and VCV matrix for a portfolio of assets and second, apply these estimates to an optimization rule that maximizes return for a given level of risk.[9] To reduce the impact of estimation error compared to using historical returns samples (DeMiguel et al., 2009b; Tu and Zhou, 2011), our study uses a multivariate probability model linking asymmetric marginals with an elliptical copula to generate simulations of future returns from which the expected return vector and VCV matrix are computed. We describe the Gaussian copula in Subsection 3.1 and marginal models that capture asymmetric volatility clustering and skewness in Subsection 3.1.1. The basic framework of MV optimization and the 18 portfolio strategies explored in our study are described in Subsection 3.2.

3.1. Dependence modeling with the Gaussian Copula

Conceptually, a copula is a multivariate distribution that combines two or more given marginal distributions into a single joint distribution.

Our work is an application of non-elliptical margins linked via an elliptical copula to reduce estimation error in MV optimization. Specifically, the Gaussian copula is selected as a parsimonious model for our setting, with a comprehensive model for our marginals that allows for autoregressive behaviour, asymmetric volatility clustering, skewness, and kurtosis as misspecified modeling of the marginals can lead to severe negative bias for elliptical correlation estimates (Fantazzini, 2009), resulting in adverse performance outcomes for MV optimization.

Sklar (1973) shows that any multivariate distribution F can be written in terms of its marginals using a copula representation where $x_1, \ldots, x_n$ are random variables and $F_1, \ldots, F_n$ are the corresponding marginal distributions as shown in equation (1):

$$F(x_1, \ldots, x_n) = C[F_1(x_1), \ldots, F_n(x_n)]. \tag{1}$$

Thus, a copula "couples" the marginal distributions to the joint distribution function F and can be re-written as shown in equation (2).

[9] A step-by-step description of the parametrization of the multivariate probability model, the sampling procedure, and the out-of-sample empirical test is detailed in Appendix A.

$$F(F_1^{-1}(x_1), \ldots, F_n^{-1}(x_n)) = C(u_1, \ldots, u_2). \tag{2}$$

The Gaussian copula belongs to the family of elliptical copulas. Its distribution function is shown in equation (3) where $\Phi_{\mathbf{R}}^n$ denotes the standardized n-variate normal distribution with correlation matrix $\mathbf{R}$.

$$C(\mathbf{u}; \mathbf{R}) = \Phi_{\mathbf{R}}^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)), \tag{3}$$

where Φ^{-1} denotes the quantile function of a univariate standard normal distribution. The corresponding copula density is given by equation (4):

$$c(\mathbf{u}; \mathbf{R}) = \frac{\frac{1}{(2\pi)^{n/2}\sqrt{|\mathbf{R}|}\exp(-0.5\boldsymbol{\zeta}'\mathbf{R}^{-1}\boldsymbol{\zeta})}}{\Pi_{j=1}^n \frac{1}{\sqrt{2\pi}}\exp(-0.5\zeta_j^2)} = \frac{\exp(-0.5\boldsymbol{\zeta}'\mathbf{R}^{-1}\boldsymbol{\zeta})}{\sqrt{\mathbf{R}}\exp(-0.5\Sigma_{j=1}^n\zeta_j^2)}, \tag{4}$$

with $\boldsymbol{\zeta} \equiv (\zeta_1, \ldots, \zeta_n)'$ and $\zeta_i = \Phi^{-1}(u_i)$ for $i = 1, \ldots, n$.

3.1.1 Marginals Modeling

Although there is no fundamental theory that suggests a distributional model for financial returns, Stoyanov et al. (2011) summarize that the extensive body of empirical research on financial returns dating from the 1950s indicates that a suitable statistical model should allow for autoregressive behaviour, volatility clustering, skewness, and fat tails (kurtosis). Thus, our mean equation uses an AR(2) model as it is shown to provide a suitably parsimonious fit for US stock and market index returns (Nelson, 1991). For the variance equation in our marginals, we use the GARCH-GJR model from Glosten et al. (1993) as it is able to capture downside asymmetric volatility clustering that is prevalent during periods of high volatility or market stress. The residuals (error distribution) are modeled using univariate standardized Skewed Student t (Skew-T) by Hansen (1994) to incorporate the effects of skewness and kurtosis. Thus our model is specified to capture asymmetries within the marginal distribution to improve the MV optimization process. Hence:

$$y_{i,t} = c_i + \sum_{j=1}^{2} \phi_{i,j} \cdot y_{i,t-j} + \epsilon_{i,t}, \text{ for } i = 1, \ldots, N, \tag{5}$$

$$\epsilon_{i,t} = h_{i,t} \cdot z_{i,t}, \tag{6}$$

$$h_{i,t}^2 = \omega_i + \alpha_i h_{i,t-1}^2 + \beta_i \epsilon_{i,t-1}^2 + \varphi_i \epsilon_{i,t-1}^2 I_{i,t-1}, \tag{7}$$

$$z_{i,t} \sim \text{standardized skewed Student } t(\nu_i, \lambda_i). \tag{8}$$

where $I_{i,t-1} = 0$ if $\epsilon_{i,t} \geq 0$ and $I_{i,t-1} = 1$ if $\epsilon_{i,t} < 0$. The skewed Student t density is given by

$$g(z|\nu, \lambda) = \begin{cases} bc\left(1 + \frac{1}{\nu-2}(\frac{bz+a}{1-\lambda})^2\right)^{-(\nu+1)/2} & z < -a/b, \\ bc\left(1 + \frac{1}{\nu-2}(\frac{bz+a}{1+\lambda})^2\right)^{-(\nu+1)/2} & z \geq -a/b. \end{cases} \tag{9}$$

The constants a, b and c are defined as

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right), b^2 = 1 + 3\lambda^2 - a^2, c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}. \tag{10}$$

During bear markets, there will be a higher probability of a large number of negative returns than positive returns. We expect this feature to be captured by a negative λ that indicates a left-skewed density.

3.2. Portfolio strategies

MV optimization (Markowitz, 1952) provides the fundamental basis for investors to have diversified portfolios and evaluate portfolio performance on the basis of risk-adjusted returns. DeMiguel et al. (2009b) compare the performance of the $1/N$ strategy versus a range of MV optimization variants that are designed to minimize estimation error. We present the basic MV framework applied by DeMiguel et al. (2009b) in Subsection 3.2.1. We apply historical and model-based estimates to a range of portfolio optimization strategies as shown in Subsection 3.2.2.

3.2.1 Mean-variance optimization framework

The weights of the chosen portfolio are given by a vector $\boldsymbol{x}_t$, that is invested in N risky assets. The investor selects $\boldsymbol{x}_t$ to maximize the expected quadratic utility function as shown in equation (11) at each time t.

$$\max_{\boldsymbol{x}_t} \boldsymbol{x}_t^\top \boldsymbol{\mu}_t - \frac{\gamma}{2}\boldsymbol{x}_t^\top \boldsymbol{\Sigma}_t \boldsymbol{x}_t, \tag{11}$$

where γ represents the investor's degree of risk aversion, $\boldsymbol{\Sigma}_t$ is the $N \times N$ variance-covariance (VCV) matrix of asset returns of the portfolio, and $\boldsymbol{\mu_t}$ is an N-dimensional vector used to denote the *expected* returns on the risky asset in excess of the risk-free rate. As the optimization problem is presented in this manner, where returns are in excess of the risk-free rate, this implicitly incorporates the constraint that the weights sum up to one. Solving (11) results in equation (12):

$$\boldsymbol{x}_t = \frac{1}{\gamma}\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t. \tag{12}$$

Similar to DeMiguel et al. (2009b), we use relative weights as opposed to absolute weights.

$$\boldsymbol{w}_t = \frac{\boldsymbol{x}_t}{|\boldsymbol{1}_N^\top \boldsymbol{x}_t|}. \tag{13}$$

Thus, $\boldsymbol{w}_t$ is a vector of *relative* portfolio weights invested at time t in N risky assets. By substituting equation (12) into equation (13), we obtain equation (14) that shows that only two inputs are required for calculation of weights for the optimized portfolio, the mean vector of expected returns ($\boldsymbol{\mu_t}$) and the VCV matrix ($\boldsymbol{\Sigma_t}$).

$$\boldsymbol{w}_t = \frac{\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t}{\boldsymbol{1}_N\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\mu}_t}. \tag{14}$$

When the mean vector and VCV matrix are calculated based upon sampling windows of historical returns, no explicit consideration is taken of asymmetries in the return distribution. In contrast, our method calculates these estimates based on Monte-Carlo simulations from the Gaussian-copula-AR(2)-GARCH-GJR-Skew-T model. Hence, we account for asymmetries within the marginals with the goal of providing a more reliable estimate of the efficient frontier.

3.2.2 List of portfolio strategies investigated

To understand the role of managing returns asymmetries in the MV framework, we compare the out-of-sample performance of the historical and model estimates when applied to a large variety of optimization rules as shown in Table 2.[10] Similar to DeMiguel et al. (2009a,b) and Tu and Zhou (2011), we assume that investors have quadratic utility preferences and a risk aversion (γ) value of 1. It should be noted that several of the optimization rules investigated are designed to be more robust to estimation error and, therefore, could partially account for return asymmetries in a latent manner.

4. Results

Our study compares the portfolio performance outcomes based on estimates of the expected return vector and VCV matrix produced by (a) sampling from a model specified to account for distributional asymmetries within portfolio returns (model-based estimates) versus (b) historical sampling windows (historical samples), for all MV optimization rules shown in Table 2. We also compare the performance of our method against the $1/N$ benchmark strategy. Given that Tu and Zhou (2011) find that longer sampling windows result in improved portfolio strategy performance, we use both $M = 120$ and $M = 240$ month sampling windows.

We follow the procedure used by Tu and Zhou (2011) and report the Sharpe Ratio and Certainty Equivalent Return (CEQ) to evaluate the out-of-sample performance of all portfolio strategies.[11] Furthermore, the performance metrics are reported after transaction costs to allow for the impact of turnover on the portfolio re-balancing process. As the international country and US industry data sets are readily investable via index futures, we apply proportional transaction costs of 1 basis point per transaction. For the US DJIA data set, that is investable directly, we apply 50 basis points per transaction. Such cost allowances are applied in prior portfolio management investigations (Balduzzi and Lynch, 1999; Tu and Zhou, 2011; Low et al., 2013)

[10]For a detailed description of each portfolio optimization strategy, see Appendix B.

[11]The equations used to calculate these metrics can be found in Tu and Zhou (2011). The CEQ is the value for a certain prospect (risk-free) that yields the same utility as the expected utility of an uncertain (risky) prospect. Large, positive values of both the Sharpe Ratio and CEQ are an indication of superior portfolio performance.

Table 2: Mean-variance optimization rules considered
This table shows a list of mean-variance optimization rules implemented within our study.

#	Model	Abbreviation
Benchmark models		
1.	1/N with re-balancing	EWR
2.	1/N without re-balancing	EWNR
3.	In-sample mean-variance (no estimation error)	MVIS
Classic approach that ignores estimation error		
4.	Sample-based mean-variance	MVS
Bayesian approaches to estimation error		
5.	Bayesian diffuse-prior	BSD
6.	Sample-based mean & adjusted VCV (Tu and Zhou, 2011)	MVTZ
7.	Bayes-Stein	BS
8.	Bayesian Data-and-Model	DM
Moment restriction approaches		
9.	Minimum-variance	MIN
10.	Missing-factor (MacKinlay and Pastor, 2000)	MP
Portfolio constraint approaches		
11.	Sample-based mean-variance with shortsale constraints	MVC
12.	Bayes-Stein with shortsale constraints	BSC
13.	Minimum-variance with shortsale constraints	MINC
14.	Minimum-variance with generalized constraints	GMINC
Combination portfolio approaches		
15.	"Three-fund" model (Kan and Zhou, 2007)	MVMIN
16.	Combination of $1/N$ and minimum-variance	EWMIN
17.	Combination of $1/N$ and Markowitz (1952) (Tu and Zhou, 2011)	EWMV
18.	Combination of $1/N$ and Kan and Zhou (2007) (Tu and Zhou, 2011)	EWKZ

The z-test of Ledoit and Wolf (2008) is applied to both the Sharpe Ratio and CEQ to indicate the statistical differences for all MV optimization rules between model-based estimates and historical samples, and model-based estimates and the $1/N$ strategy. The z-test of Ledoit and Wolf (2008) is used in several prior empirical studies in portfolio optimization (Fletcher, 2011; Disatnik and Katz, 2012; DeMiguel et al., 2013a). In our application of the z-test, similar to Fletcher (2011), the test statistics are corrected for the effects of heteroscedasticity and serial correlation using an automatic lag selection, without pre-whitening, using the method of Newey and West (1994).

For combination portfolios, we observe how using model-based estimates alters the weights applied to the optimization component compared to the target component. Tu and Zhou (2011) show that an increase in weight applied to the optimization component is indicative of the increase in informativeness of the returns sample applied to the combination portfolio rule.

4.1. International country setting (9 indices)

Table 3 shows the Sharpe Ratio, CEQ, and the z-test of Ledoit and Wolf (2008) for a data set of 9 assets consisting of international country indices. In the benchmark models shown in Panel A, a large degree of estimation error is evident when we compare the Sharpe Ratio and CEQ values of in-sample mean-variance (MVIS) that has no estimation error versus both $1/N$ with rebalancing (EWR) and $1/N$ without rebalancing (EWNR). The Sharpe Ratio of MVIS is more than twice the magnitude of that produced by the main benchmark portfolio EWR and the CEQ is more than five times larger.

Panel B displays results for alternative approaches using the 120 month window. We see that statistically higher Sharpe Ratios and CEQ are produced for 7 (8) of the portfolio optimization rules when model-based estimates are applied as opposed to their historical sample counterparts, indicating superior performance. When historical samples are used, only the minimum-variance with generalized constraints (GMINC) portfolio produces a higher Sharpe Ratio and CEQ than the $1/N$ strategy. Once the model-based estimates are applied, the sample-based mean-variance with shortsale constraints (MVC), minimum-variance with shortsale constraints (MINC), Bayes-Stein with shortsale constraints (BSC), and GMINC are improved to the point of outperforming the $1/N$ strategy in terms of the CEQ. The same applies for the Sharpe Ratio except for the BSC method. However, none of these performance improvements are statistically different from the $1/N$ strategy.

In Panel C, where the longer sampling window of 240 months is used, applying model-based estimates improves the Sharpe Ratio and CEQ values beyond the use of historical samples for 13 strategies (the exceptions being MacKinlay and Pastor (2000) missing-factor model (MP) and 1-factor data-and-model (DM1)). Of these 13 strategies, the Sharpe Ratios (CEQ) are statistically different and in favour of the model-based estimates version for 10 (9) of them. Seven of the MV rules (MIN, MINC, GMINC, Kan and Zhou (2007) 'three-fund' model (MVMIN), mixture of minimum-variance and $1/N$ (EWMIN), combination of $1/N$ and Kan and Zhou (2007) (EWKZ), and combination of $1/N$ and Markowitz (1952) (EWMV)) produce higher Sharpe Ratios and CEQ than the $1/N$ strategy, although none are statistically different. The GMINC is the only strategy to outperform the $1/N$ both when historical samples and model-based estimates are used. Thus, our results are similar to that of Tu and Zhou (2011) who report improved performance when longer historical sampling windows are used. We find this result still holds when we use longer sampling windows to parameterize our model-based estimates as they continue to outperform longer historical sampling windows.

Scrutinizing the range of models that are enhanced, we find that accounting for returns asymmetries does improve the estimates of the expected means and VCV matrix. For example, in the $M = 120$ case, strategies sample-based mean-variance (MVS), sample-based mean & adjusted VCV developed by Tu and Zhou (2011) (MVTZ), and Bayesian diffuse-prior (BSD)[12] show an improvement from a Sharpe Ratio of -0.0605 to 0.0695. However, the difficulty in estimating the vector of expected returns is such that ignoring the expectation and focusing on strategies that rely on estimates of the VCV matrix (e.g., MIN, GMINC, MINC) only, produces a better outcome, as

[12] As described in Subsection Appendix B.1.1, the scaling effect imposed upon the VCV matrix for both the MVTZ and BSD is mathematically significant for applications with short sampling windows. Therefore, as we use long sampling windows of 120 and 240 months, the effects upon the portfolio weights are negligible and the resulting portfolio outcomes for the MVTZ and BSD strategies are virtually indistinguishable from the MVS strategy. This result continues to persist throughout the US industry and US DJIA data sets explored within our study.

Table 3: Portfolio performance results across optimization rules - international country indices, (N=9) This table shows the Sharpe Ratio and CEQ (*100) metric for various portfolio rules (see Table 2) applied to the international country portfolio data set when historical samples (e.g., SR_H, CEQ_H) or model-based estimates (e.g., SR_M, CEQ_M) are applied. The z-test of (Ledoit and Wolf, 2008) is reported to show that the performance metrics are significantly different between model-based estimates versus historical samples (e.g., $z_{SR_{M\neq H}}$, $z_{CEQ_{M\neq H}}$), and the $1/N$ strategy (e.g., $z_{SR_{M\neq 1/N}}$, $z_{CEQ_{M\neq 1/N}}$). We use the method of Newey and West (1994) to correct the test statistics for the effects of serial correlation and heteroscedasticity using an automatic lag selection without pre-whitening. Panel A contains the benchmark models. Panel B and Panel C show the performance of the portfolio rules for sample window lengths of 120 and 240 months, respectively.

Portfolio strategy	Sharpe Ratio (SR)	Certainty Equivalent (CEQ)
Panel A: Benchmark models		
EWR (1/N)	0.0913	0.3275
EWNR	0.0931	0.3366
MVIS	0.2194	1.6490

Portfolio strategy	SR_H	SR_M	$z_{SR_{M\neq H}}$	$z_{SR_{M\neq 1/N}}$	CEQ_H	CEQ_M	$z_{CEQ_{M\neq H}}$	$z_{CEQ_{M\neq 1/N}}$
Panel B: Alternative Portfolio Rules, Window size $M = 120$								
Classic approach that ignores estimation error								
MVS	-0.0605	0.0695	2.4589^{b}	1.5816	-37.7824	-7.9081	2.5826^{b}	1.7395^{a}
Bayesian approach to estimation error								
MVTZ	-0.0605	0.0695	2.4589^{b}	1.5816	-37.7824	-7.9081	2.5826^{b}	1.7395^{a}
BSD	-0.0605	0.0695	2.4589^{b}	1.5816	-37.7824	-7.9081	2.5826^{b}	1.7395^{a}
BS	-0.0610	-0.0496	0.9734	1.9795^{b}	-13.6983	-1.1451	2.1783^{b}	1.3192^{a}
DM1	0.0814	0.0683	1.1616	1.6887^{a}	0.2735	-8.6981	1.9807^{b}	1.8917^{a}
Moment restrictions								
MP	0.0846	0.0462	1.9971^{b}	1.7851^{a}	0.2989	0.1054	1.0053	1.0269
MIN	0.0716	0.0794	0.8720	1.6412	0.2142	0.2955	0.3369	0.1886
Portfolio constraints								
MVC	0.0494	0.0960	1.9803^{b}	0.2038	0.1105	0.3816	1.8457^{a}	0.4093
BSC	0.0619	0.0885	1.6837^{a}	0.1198	0.1903	0.3319	1.7459^{a}	0.0335
MINC	0.0912	0.0964	0.5373	0.2170	0.3059	0.3860	0.4234	0.4894
GMINC	0.0973	0.0995	0.3857	0.5466	0.3409	0.3852	0.3375	0.7177
Combination portfolios								
MVMIN	-0.0611	0.0668	2.3077^{b}	1.6946^{a}	-7.8013	-0.0904	2.0680^{b}	1.6648^{a}
EWMIN	0.0757	0.0687	1.6922^{a}	1.6806^{a}	0.2349	0.2291	0.0305	0.6692
EWKZ	0.0149	-0.0646	2.1036^{b}	2.0270^{b}	-0.0995	-6.8459	2.0105^{b}	1.8111^{a}
EWMV	0.0129	0.0719	1.9943^{b}	1.3788	-0.5616	0.2541	1.9757^{b}	0.4067
Panel C: Alternative Portfolio Rules, Window size $M = 240$								
Classic approach that ignores estimation error								
MVS	0.0397	0.0511	1.7218^{a}	1.8278^{a}	0.0371	0.1223	1.7036^{a}	1.9177^{a}
Bayesian approach to estimation error								
MVTZ	0.0397	0.0511	1.7218^{a}	1.8278^{a}	0.0371	0.1223	1.7036^{a}	1.9177^{a}
BSD	0.0397	0.0511	1.7218^{a}	1.8278^{a}	0.0371	0.1223	1.7036^{a}	1.9177^{a}
BS	0.0671	0.0725	1.1434	1.2846	0.2099	0.2542	0.2126	0.9173
DM1	0.0603	0.0515	1.2113^{a}	1.8193^{a}	0.1714	0.1246	0.9882	1.9081^{a}
Moment restrictions								
MP	0.0898	0.0852	0.8909	1.3122	0.3235	0.2928	1.4758	1.6485
MIN	0.0808	0.0988	1.6749^{a}	0.4750	0.2574	0.4086	0.8645	1.7564^{a}
Portfolio constraints								
MVC	0.0267	0.0708	1.9801^{a}	1.2375	-0.0155	0.2387	2.0575^{b}	1.7772^{a}
BSC	0.0508	0.0572	0.8727	1.6730^{a}	0.1282	0.1632	0.1740	1.8182^{a}
MINC	0.0905	0.1000	1.6518^{a}	0.5449	0.3040	0.4086	1.6692^{a}	1.7838^{a}
GMINC	0.0957	0.0958	0.0102	0.4401	0.3339	0.3705	0.6639	0.6793
Combination portfolios								
MVMIN	0.0752	0.0933	1.6679^{a}	0.1186	0.2414	0.3742	1.7235^{a}	0.8286
EWMIN	0.0877	0.1044	1.8719^{a}	0.8496	0.2927	0.4282	2.0145^{b}	1.6546^{a}
EWKZ	0.0793	0.0961	1.6965^{a}	0.2930	0.2607	0.3820	1.7793^{a}	0.7375
EWMV	0.0651	0.0925	1.8706^{a}	0.0642	0.2014	0.3502	1.7073^{a}	0.5198

^{a,b} indicates that the two performance metrics under comparison are statistically different at the 10 and 5 percent level, respectively.

evident in both Panels B and C. The other strategies that show strong performance improvements are the combination portfolios proposed by Tu and Zhou (2011). Combination portfolios are an alternative solution to ignoring poor estimates of the mean as they shift the portfolio weights towards the $1/N$ when the input sample estimates are too unreliable for the optimization component to truly maximize the investor's utility. We find that the performance improvements across the MV optimization rules using model-based estimates are insufficient to produce superior and statistically different outcomes compared to the $1/N$ strategy due to higher turnover and transaction costs.

4.2. US industry setting (17 indices)

Table 4 shows the Sharpe Ratio, CEQ and z-tests of Ledoit and Wolf (2008) for a data set of 17 US industry indices. In Panel A, we observe that the in-sample benchmark MV strategy with no estimation error produces a Sharpe Ratio and CEQ of 0.2313 and 1.1820, respectively. These results are much higher than the $1/N$ strategy or the MVS (Panel B) that produce Sharpe Ratios of 0.1218 and -0.0407, respectively. This indicates that there is a large magnitude of estimation error in MV optimization for the 17 US industry data set.

In Panel B ($M = 120$), when applying historical sampling windows, we find that the only strategies that outperform the $1/N$ are the MINC and GMINC rules in terms of the Sharpe Ratio. Using model-based estimates improves 12 of the MV rules compared to historical samples as evaluated by either the Sharpe Ratio or CEQ. Of these 12 strategies, we find that 10 show statistical superiority. The only rules that are not improved are the MVC, BSC and DM portfolio strategies. Of the improved portfolio rules, seven are improved beyond the $1/N$ for both the Sharpe Ratio and CEQ, of which five are statistically superior. These strategies are variants of the MIN rule, and the combination portfolios.

In Panel C ($M = 240$), we find that longer historical sampling results in higher Sharpe Ratio and CEQ values for all strategies. Specifically, the MVS, MVTZ, BSD, Bayes-Stein (BS), MVMIN, and EWKZ rules that exhibit negative Sharpe Ratios in Panel B are now positive in Panel C. In addition, the MINC and the GMINC now outperform the $1/N$ in both the Sharpe Ratio and CEQ. Similar to Panel B, using model-based estimates enhances 12 of the optimization rules in both the Sharpe Ratio and CEQ (the exceptions being MINC, GMINC and DM approaches) where 8 of these are statistically superior. Out of these improved portfolio rules, none are statistically superior to the $1/N$.

We find that similar to the other data sets, using longer historical windows of 240 months as inputs into the portfolio rules continues to provide a much better outcome than using 120 months. However, using longer historical windows to parameterize our model-based estimates does not lead to as much improvement in the performance of the portfolio strategies compared to the smaller-dimensioned portfolios investigated.

Table 4: Portfolio performance results across optimization rules - US industry indices, (N=17)
This table shows the Sharpe Ratio and CEQ (*100) metric for various portfolio rules (see Table 2) applied to the US industry portfolio data set when historical samples (e.g., SR_H, CEQ_H) or model-based estimates (e.g., SR_M, CEQ_M) are applied. The z-test of (Ledoit and Wolf, 2008) is reported to show that the performance metrics are significantly different between model-based estimates versus historical samples (e.g., $z_{SR_{M\neq H}}$, $z_{CEQ_{M\neq H}}$), and the $1/N$ strategy (e.g., $z_{SR_{M\neq 1/N}}$, $z_{CEQ_{M\neq 1/N}}$). We use the method of Newey and West (1994) to correct the test statistics for the effects of serial correlation and heteroscedasticity using an automatic lag selection without pre-whitening. Panel A contains the benchmark models. Panel B and Panel C show the performance of the portfolio rules for sample window lengths of 120 and 240 months, respectively.

Portfolio strategy	Sharpe Ratio (SR)	Certainty Equivalent (CEQ)
	Panel A: Benchmark models	
EWR (1/N)	0.1218	0.4723
EWNR	0.1145	0.4285
MVIS	0.2313	1.1820

Portfolio strategy	SR_H	SR_M	$z_{SR_{M\neq H}}$	$z_{SR_{M\neq 1/N}}$	CEQ_H	CEQ_M	$z_{CEQ_{M\neq H}}$	$z_{CEQ_{M\neq 1/N}}$
			Panel B: Alternative Portfolio Rules, Window size $M = 120$					
Classic approach that ignores estimation error								
MVS	-0.0407	0.0916	2.1905[b]	1.6630[a]	-54.5268	-4.0672	2.3605[b]	1.8557[a]
Bayesian approach to estimation error								
MVTZ	-0.0407	0.0916	2.1905[b]	1.6630[a]	-54.5268	-4.0672	2.3605[b]	1.8557[a]
BSD	-0.0407	0.0916	2.1905[b]	1.6630[a]	-54.5268	-4.0672	2.3605[b]	1.8557[a]
BS	-0.0377	0.1248	3.1328[c]	0.0527	-9.4469	0.7087	2.6368[c]	0.5108
DM1	0.1084	0.0880	1.2824	1.4760	0.4124	-4.0290	1.3259	1.8406
DM3	0.1056	0.0761	1.4032	1.7596[a]	0.3928	-3.8960	1.4429	1.9956[b]
DM4	0.1047	0.0738	1.4368	1.7733[a]	0.3854	-4.7768	1.3882	1.9793[b]
Moment restrictions								
MP	0.0563	0.0696	0.3891	1.6660[a]	0.1572	0.2269	0.3947	1.5571
MIN	0.0801	0.1659	3.4789[c]	2.0933[b]	0.2321	0.6746	4.0599[c]	2.0723[b]
Portfolio constraints								
MVC	0.1057	0.0672	1.9286[a]	2.0992[b]	0.4815	0.2211	1.9866[a]	1.6231[a]
BSC	0.1004	0.0827	1.4557	1.6912[a]	0.4188	0.2980	0.5464	1.3581
MINC	0.1276	0.1516	1.6827[a]	2.3378[b]	0.4179	0.5990	1.9127[a]	2.1111[b]
GMINC	0.1327	0.1368	0.6387	1.7692[a]	0.4540	0.5332	1.3546	1.6538[a]
Combination portfolios								
MVMIN	-0.0399	0.1355	3.5291[c]	0.3148	-1.6812	0.6305	2.8655[c]	0.6095
EWMIN	0.0992	0.1603	3.4470[c]	2.2717[b]	0.3066	0.6389	4.2347[c]	2.2121[b]
EWKZ	-0.0344	0.1209	2.9909[c]	0.0269	-2.9660	0.5345	2.7753[c]	0.3224
EWMV	0.0259	0.1686	4.7862[c]	2.1493[b]	-0.0699	0.7390	2.8797[c]	2.0833[b]
			Panel C: Alternative Portfolio Rules, Window size $M = 240$					
Classic approach that ignores estimation error								
MVS	0.0240	0.0682	1.7310[a]	1.9115[a]	-0.4452	0.2319	2.0739[b]	1.6906[a]
Bayesian approach to estimation error								
MVTZ	0.0240	0.0682	1.7310[a]	1.9115[a]	-0.4452	0.2319	2.0739[b]	1.6906[a]
BSD	0.0240	0.0682	1.7310[a]	1.9115[a]	-0.4452	0.2319	2.0739[b]	1.6906[a]
BS	0.0526	0.0628	0.5802	1.7471[a]	0.1343	0.2198	1.7441[a]	1.7124[a]
DM1	0.1071	0.0690	1.7360[a]	1.8870[a]	0.3844	0.2374	0.6917	1.8595[a]
DM3	0.1033	0.0785	1.4688	1.6771[a]	0.3712	0.3017	0.3351	1.2135
DM4	0.1066	0.0778	1.3902	1.6657[a]	0.3886	0.2969	0.4473	1.2419
Moment restrictions								
MP	0.1168	0.1184	0.5624	0.9642	0.4617	0.4693	0.4954	0.1902
MIN	0.1110	0.1260	0.4801	0.2063	0.3583	0.4810	0.8990	0.0942
Portfolio constraints								
MVC	0.0585	0.1136	1.7182[a]	0.3129	0.1703	0.4632	1.9875[a]	0.0638
BSC	0.0874	0.1301	1.7271[a]	0.5206	0.3163	0.5400	1.7762[a]	0.8232
MINC	0.1414	0.1221	0.9306	0.0192	0.4806	0.4595	0.5167	0.1904
GMINC	0.1397	0.1216	1.2787	0.0202	0.4871	0.4618	0.6713	0.2218
Combination portfolios								
MVMIN	0.0835	0.1224	1.7720[a]	0.0302	0.2664	0.4640	1.7334[a]	0.0991
EWMIN	0.1165	0.1297	0.5388	0.5050	0.3787	0.4949	1.3807	0.3221
EWKZ	0.0948	0.1151	1.6572[a]	0.4356	0.3049	0.4282	1.4605	0.6426
EWMV	0.0595	0.1068	1.7544[a]	0.9859	0.1756	0.4093	1.7011[a]	0.8378

[a,b,c] indicates that the two performance metrics under comparison are statistically different at the 10, 5, and 1 percent level, respectively.

4.3. US DJIA setting (30 stocks)

Table 5 shows a similar Sharpe Ratio and CEQ analysis of the US DJIA stocks. As shown in Panel A, we find that the $1/N$ strategy performs poorly as it results in a Sharpe Ratio of -0.8405 and a CEQ of -0.4080, whereas the MV strategies produce positive Sharpe Ratios and CEQ both when historical and model-based estimates are applied. The Sharpe Ratios from MVIS are almost twice as high as the Sharpe Ratios produced when model-based estimates or historical samples are applied to the MV optimization rules.

In Panel B ($M = 120$), we find that 10 of the optimization rules show increased Sharpe Ratios with the application of model-based estimates compared to historical samples. Moreover, 7 of these strategies exhibit statistically significant improvements in the Sharpe Ratio. Notably, we find that application of model-based estimates results in 14 MV rules producing higher Sharpe Ratios compared to the $1/N$ strategy, with 11 of these strategies being statistically superior. Similar results are found for the CEQ metric where 11 strategies result in statistically significant out performance when model-based estimates are applied compared to both historical samples and the $1/N$ strategy.

In Panel C ($M = 240$), the magnitude of improvements in the Sharpe Ratios and CEQ for each strategy when model-based estimates are applied in relation to historical samples are similar to Panel B. However, a greater number of MV strategies are improved when model-based estimates are applied. We observe that 12 MV strategies exhibit Sharpe Ratios that are greater and statistically different when comparing the use of model-based estimates to historical samples. In addition, compared to the $1/N$ strategy, use of model-based estimates results in greater and statistically different Sharpe Ratios for 11 MV optimization rules.

We apply our investigation to the US DJIA data set as the distributional characteristics of stocks are different from indices, and re-balancing a portfolio of stocks requires higher transaction costs. As portfolio optimization has higher turnover requirements to implement compared to the $1/N$ strategy, other things being equal, application of higher transaction costs biases the portfolio optimization to favour the $1/N$ strategy. Nevertheless, we find that the main difference between the US DJIA data set compared to the previous data sets is that the $1/N$ strategy and the MV optimization rules with short sales constraints perform poorly and produce similarly low Sharpe Ratios and CEQ.

Both the $1/N$ strategy and the short sales constrained MV rules are long-only strategies that are unable to profit from downward stock market movements by short-selling. In contrast, the other portfolio strategies have the flexibility of allowing for short sales or negative portfolio weights. Allowing for short sales is particularly crucial for investment portfolios that exhibit left tail dependence (Patton, 2004). Intuitively, during a financial crisis, increases in asset correlations result in all assets experiencing negative returns, therefore in such circumstances any long positions will result in losses and only short positions will result in positive returns.

Table 5: Portfolio performance results across optimization rules - US DJIA Stocks, (N=30)
This table shows the Sharpe Ratio and CEQ (*100) metric for various portfolio rules (see Table 2) applied to the US DJIA portfolio data set when historical samples (e.g., SR_H, CEQ_H) or model-based estimates (e.g., SR_M, CEQ_M) are applied. The z-test of (Ledoit and Wolf, 2008) is reported to show that the performance metrics are significantly different between model-based estimates versus historical samples (e.g., $z_{SR_{M\neq H}}$, $z_{CEQ_{M\neq H}}$), and the $1/N$ strategy (e.g., $z_{SR_{M\neq 1/N}}$, $z_{CEQ_{M\neq 1/N}}$). We use the method of Newey and West (1994) to correct the test statistics for the effects of serial correlation and heteroscedasticity using an automatic lag selection without pre-whitening. Panel A contains the benchmark models. Panel B and Panel C show the performance of the portfolio rules for sample window lengths of 120 and 240 months, respectively.

Portfolio strategy	Sharpe Ratio (SR)	Certainty Equivalent (CEQ)
Panel A: Benchmark models		
EWR (1/N)	-0.8405	-0.4080
EWNR	-0.9132	-0.4206
MVIS	1.3458	0.4446

Portfolio strategy	SR_H	SR_M	$z_{SR_{M\neq H}}$	$z_{SR_{M\neq 1/N}}$	CEQ_H	CEQ_M	$z_{CEQ_{M\neq H}}$	$z_{CEQ_{M\neq 1/N}}$
Panel B: Alternative Portfolio Rules, Window size $M = 120$								
Classic approach that ignores estimation error								
MVS	0.5173	0.6295	3.3427^c	4.4121^c	0.3977	0.4010	3.4208^c	4.6820^c
Bayesian approach to estimation error								
MVTZ	0.5173	0.6295	3.3427^c	4.4121^c	0.3977	0.4010	3.4208^c	4.6820^c
BSD	0.5173	0.6295	3.3427^c	4.4121^c	0.3977	0.4010	3.4208^c	4.6820^c
BS	0.6157	0.5135	4.1101^c	4.1031^c	0.4244	0.3989	4.2612^c	4.0721^c
DM1	0.5909	0.5896	2.8180^c	4.4296^c	0.3812	0.3855	2.8345^c	4.4285^c
DM3	0.5474	0.5905	2.7968^c	4.2475^c	0.3822	0.3855	2.8162^c	4.2657^c
DM4	0.5588	0.5968	2.7980^c	4.3801^c	0.3806	0.3884	2.8176^c	4.4167^c
Moment restrictions								
MP	0.7878	0.5511	4.7336^c	4.6633^c	0.3834	0.2884	4.6247^c	4.7572^c
MIN	-0.7247	-0.6047	0.6138	0.8275	-0.4960	-0.4453	0.6342	0.9299
Portfolio constraints								
MVC	-0.2870	-0.8764	3.3155^c	0.2322	-0.4000	-0.4245	3.3179^c	0.0749
BSC	-0.8133	-0.7877	0.1749	0.4090	-0.4327	-0.4009	0.1969	0.6017
MINC	-0.8488	-0.8405	1.5086	0.0016	-0.4094	-0.4124	1.4878	1.1682
GMINC	-0.8408	-0.9577	1.7905^a	1.9396^b	-0.4080	-0.4131	1.7529^a	1.8133^a
Combination portfolios								
MVMIN	0.6403	0.5777	4.4040^c	4.3239^c	0.3925	0.4325	4.5794^c	4.4664^c
EWMIN	-1.0136	-0.7590	3.0466^c	0.3093	-0.4502	-0.4462	3.0008^c	0.4123
EWKZ	0.6402	0.5753	4.4041^c	4.5421^c	0.3906	0.4321	4.5795^c	4.2922^c
EWMV	0.5170	0.6267	3.3427^c	4.6622^c	0.3955	0.4010	3.4208^c	4.5890^c
Panel C: Alternative Portfolio Rules, Window size $M = 240$								
Classic approach that ignores estimation error								
MVS	0.5412	0.6077	3.1178^c	4.0542^c	0.3788	0.3846	3.1786^c	4.0518^c
Bayesian approach to estimation error								
MVTZ	0.5412	0.6077	3.1178^c	4.0542^c	0.3788	0.3846	3.1786^c	4.0518^c
BSD	0.5412	0.6077	3.1178^c	4.0542^c	0.3788	0.3846	3.1786^c	4.0518^c
BS	0.6725	0.4922	4.0690^c	3.2200^c	0.4043	0.3759	4.2806^c	3.2128^c
DM1	0.5935	0.6077	2.9372^c	4.0016^c	0.3784	0.3788	2.9881^c	4.0506^c
DM3	0.5217	0.6086	2.7329^c	4.0630^c	0.3762	0.3791	2.7510^c	4.0608^c
DM4	0.5274	0.6096	2.7615^c	4.0714^c	0.3774	0.3792	2.7811^c	4.0695^c
Moment restrictions								
MP	0.9220	0.8333	4.5040^c	3.7265^c	0.3978	0.3885	4.6031^c	3.7718^c
MIN	-0.8014	-0.7570	0.5236	0.5512	-0.4554	-0.4333	0.5496	0.7876
Portfolio constraints								
MVC	-0.8758	-0.9669	0.2122	0.7235	-0.4293	-0.4556	0.2075	0.4878
BSC	-0.9889	-0.9644	1.2737	0.7145	-0.4296	-0.4554	1.2580	0.4791
MINC	-0.8900	-0.8422	3.2456^c	0.8491	-0.4111	-0.4085	3.2357^c	0.0762
GMINC	-0.8411	-0.8383	2.0939^b	1.2894	-0.4081	-0.4073	2.0905^b	1.3504
Combination portfolios								
MVMIN	0.6433	0.7220	4.8027^c	4.5506^c	0.3818	0.4100	4.8621^c	4.6126^c
EWMIN	-0.9308	-0.8048	1.7577^a	0.3146	-0.4323	-0.4280	1.7323^a	0.5610
EWKZ	0.5338	0.7631	4.8028^c	4.5506^c	0.3712	0.4211	4.8622^c	4.6126^c
EWMV	0.5411	0.6076	3.1179^c	4.0543^c	0.3788	0.3846	3.1786^c	4.0518^c

[a,b,c] indicates that the two performance metrics under comparison are statistically different at the 10, 5, and 1 percent level, respectively.

4.4. Discussion

Across all three data sets investigated, we find that use of model-based estimates for both sampling windows of 120 and 240 months generally results in Sharpe Ratio and CEQ values that are higher and statistically superior compared to estimates directly based on historical samples. Use of longer historical samples for the model-based approach often increases the number of MV optimization strategies that are improved. Across the data sets, for $M = 120$ months, the MV strategies that exhibit improvements with the application of the model-based approach are the MVS, MVTZ, BSD, MIN, MINC, and EWMV. For longer sampling windows of $M = 240$ months, this extends to include the BSC, MVMIN, EWMIN, and EWKZ rules. We generally find that the best performing MV optimization rules are variants of the MIN strategy and the combination portfolio rules of Tu and Zhou (2011). Model-based estimates also show superior performance if compared to the base-case MV model.

We find that application of the model-based approach to MV optimization with the goal of outperforming the $1/N$ strategy remains an elusive challenge. Although there are instances where several strategies (e.g., MIN, MVMIN, GMINC, BS, BSC, EWMIN, EWMV) produce higher Sharpe Ratios than the $1/N$ in two out of three of the data sets investigated, it is difficult to show both superior and statistical improvement beyond the $1/N$ universally. Only the DJIA data set reveals evidence that variants of the MV optimization rules outperforming the $1/N$ in a statistically significant manner. However, this is likely due to the characteristics of stocks within the DJIA being representative of large-cap US stocks, producing a less diversified portfolio compared to the international country indices and US industry data sets. Thus, as the DJIA data set is likely to exhibit greater asymmetric correlations, strategies that are long-only (e.g., $1/N$) perform poorly.

Our results show that improvement in portfolio optimization strategies based on model-based estimates are stronger for US datasets, and less so for the international equities dataset. This is potentially due to the international equities dataset being a more diversified portfolio that exhibits less asymmetries compared to the US datasets.

In comparison to the $1/N$ strategy, portfolio optimization requires a much higher turnover to implement (DeMiguel et al., 2009b). Thus, the benefits to be gained by using model-based estimates in MV optimization, with varying force, are outweighed by the impact of transaction costs when compared to the $1/N$ strategy.[13]

4.5. Coefficients of the combination rules

The analysis in Subsections 4.1, 4.2, and 4.3 indicate that the best performing strategies are variants of the MIN rule and the combination rules. Combination rules are considered to be shrinkage estimators (Tu and Zhou, 2011) that either have the $1/N$ (e.g., EWKZ, EWMV, EWMIN) or the

[13] Additional analysis on the re-balancing weights of each portfolio strategy are given in Appendix C.

MIN as shrinkage targets (e.g., MVMIN). The degree of shrinkage applied is a tradeoff between bias and variance (Jorion, 1986) as represented by the target and optimization components, respectively. For example, in the EWMV strategy, the $1/N$ rule is biased and has zero variance whereas the MV optimization rule is asymptotically unbiased, but can exhibit large variance when small samples or unsophisticated estimates of expected returns are applied. An increase in the weighting of the $1/N$ component increases the bias and decreases the variance, thus the performance of the combination rule is a balance between the bias and variance. Tu and Zhou (2011) show that as the size of the sampling window and the reliability of the estimate of expected returns increases, so does the coefficient of the optimization component.

Intuitively, as the target portfolios make very little or no use of sampling information and the optimization components[14] are dependent upon the estimate of expected returns to calculate optimal allocations, it is expected that if the estimates are informative, the optimization component will have a larger weighting compared to the target component. Thus, generally, a larger weighting on the optimization component (α) can be interpreted as a greater level of confidence in method used in the estimation of expected returns.

In Table 6, we report the mean and standard errors of the coefficient on the optimization component (α) as shown by equation B.1 when estimates based on historical samples and our model-based approach are applied. We test whether the difference between α values produced by the two approaches is significantly different from zero using the two-sided, non-parametric Wilcoxon rank-sum test at the 5% level (Wilcoxon, 1945; Siegel, 1956). To assess whether the use of model-based estimates are able to simultaneously improve portfolio performance and increase the coefficients of the optimized components, we identify cases where the portfolio performance is superior to that based on the historical samples ("#"). The portfolio performance is evaluated using the CEQ measure.

For a window size of 120 months, across all 3 panels, using model-based estimates compared to historical samples increases the mean coefficient applied to the optimization component and is significant at the 5% level. For example, in Panel A, we can see that the average optimized component weighting, KZ, in the EWKZ strategy increases from 53.08% to 73.20% when model-based estimates are used and in Panel C, for the EWMV rule, the MV component increases from 67.73% to 74.17%. The greater weight applied to the optimized components, when model-based estimates are used instead of historical samples, leads to a superior performance of the combination portfolios as can be seen for 10 of the optimal combination rules across all data sets investigated (e.g., Panel A - MVMIN, EWMV; Panel B - MVMIN, EWMIN, EWKZ, EWMV; Panel C - MVMIN,

[14]The optimized component is defined as the strategy that has a greater reliance on inputs from the sample estimates. For example, for the MVMIN rule, the optimal rule would be the MVS as it requires both the estimated mean and VCV matrix from the sample estimates compared to the MIN that ignores the mean and only uses the VCV matrix.

Table 6: Coefficients applied to optimization components of the optimal combination rules
This table shows the descriptive statistics for the coefficients applied to the optimization components of the optimal combination rules. The optimization components are denoted in bold. For example, EW**MIN**, the reported statistics are for the weights applied to the minimum variance strategy, being the optimized component of that combination rule. See Table 2 for definitions of the rules. The mean of the weights are presented as percentage points.

Portfolio strategy	M=120				M=240			
	Hist. samples		Model-based estimates		Hist. samples		Model-based estimates	
	Mean	Std. Err.	Mean	Std. Err.	Mean	Std. Err.	Mean	Std. Err.
Panel A: International Country Indices, N=9								
MVMIN	23.33	12.86	30.65$^{\#,*}$	33.72	12.50	5.83	10.88$^{\#,*}$	10.39
EW**MIN**	0.19	0.17	11.44*	30.22	0.37	0.13	0.44$^{\#}$	0.34
EW**KZ**	53.08	9.47	73.20*	25.33	48.14	9.34	56.88$^{\#,*}$	21.18
EW**MV**	21.14	14.98	49.23$^{\#,*}$	39.93	11.55	10.71	24.45$^{\#,*}$	34.02
Panel B: US Industry Indices Setting, N=17								
MVMIN	14.97	5.67	18.03$^{\#,*}$	30.71	15.46	5.77	10.07$^{\#,*}$	16.63
EW**MIN**	0.22	0.07	6.67$^{\#,*}$	22.48	0.48	0.12	0.42$^{\#}$	0.30
EW**KZ**	48.62	8.76	66.28$^{\#,*}$	27.69	48.35	6.42	48.67$^{\#}$	19.55
EW**MV**	10.82	5.21	27.26$^{\#,*}$	35.53	12.89	6.84	13.55$^{\#}$	25.43
Panel C: US DJIA Stocks, N=30								
MVMIN	3.86	1.49	27.80$^{\#,*}$	31.29	2.39	0.71	41.26$^{\#,*}$	24.90
EW**MIN**	0.32	0.08	16.95$^{\#,*}$	35.57	0.45	0.09	0.11$^{\#}$	2.18
EW**KZ**	94.63	3.06	98.45$^{\#,*}$	1.64	96.54	0.90	99.05$^{\#,*}$	0.79
EW**MV**	67.73	3.55	74.17$^{\#,*}$	8.41	82.89	1.05	85.23$^{\#,*}$	1.09

[#] indicates a higher CEQ value when model-based estimates are applied compared to historical samples.
[*] indicate that the differences between the mean of the optimal components given by historical samples and asymmetric estimates are significantly different from zero using the two-sided, non-parametric Wilcoxon rank-sum test at the 5% level.

EWMIN, EWKZ, EWMV).

For the longer sampling window of 240 months, in Panel A, when model-based estimates are applied, the EWKZ and EWMV strategies exhibit higher coefficients (significant at the 5% level) for the optimization components. Notably, across all panels of the $M = 240$ case, 4 strategies exhibit lower coefficients for the optimization component (significant at the 5% level) when model-based estimates are used (Panel A - MVMIN ; Panel B - MVMIN, EWMIN ; Panel C - EWMIN). For example, in Panel B, when model-based estimates are used, the optimized component of MV in the MVMIN rule decreases from 15.46% to 10.07%. However, these strategies are improved in terms of their CEQ when model-based estimates are applied compared to historical samples. This suggests that the true optimal rules for these cases are closer to the target component.

We observe that for $M = 120$, the increase in coefficients for the optimization component ranges between 3-28%. Therefore, application of model-based estimates for $M = 120$ shows greater improvements that are supported by the improved CEQ values in comparison to historical samples. For $M = 240$, the application of model-based estimates 'fine-tunes' the optimization procedure as the standard errors are smaller. Therefore, the improvement in using model-based estimates is less pronounced when longer sampling windows are used. Using longer sampling windows for historical samples and model-based estimates decreases the standard errors in the combination coefficients of the optimal combination portfolio rules.

Overall, we find that model-based estimates are more reliable and informative compared to historical samples, thus improving the combination portfolio rules by increasing the weighting on the optimization component. This effect is less pronounced for the longer sampling windows of 240 months.

5. Conclusion

For decades, mean-variance (MV) optimization by Markowitz (1952) has been taught across business schools globally and is widely used in industry. Despite this dual popularity, DeMiguel et al. (2009b) investigating the performance of several advances in MV optimization, find that none of these advances consistently outperform the naïve equally-weighted ($1/N$) portfolio in terms of the Sharpe Ratio and Certainty Equivalent Return (CEQ), for a range of data sets. For academics and practitioners alike, these findings are troublesome and have intriguing implications regarding the empirical applications of portfolio optimization and modern portfolio theory.

DeMiguel et al. (2009b) use rolling-sampling windows of historical returns which might not be very informative or reliable, and do not account for asymmetries in the returns distribution. Notably, Markowitz (1952) explicitly recommends the use of a probability model to generate the inputs required by the MV model. Stoyanov et al. (2011) document that the extensive body of empirical research on financial returns dating from the 1950s indicates that a suitable model should allow for autoregressive behaviour, volatility clustering, skewness, and kurtosis. Furthermore, Fantazzini (2009) finds that when the financial returns data exhibit asymmetries such as skewness, and symmetric marginals are applied, the estimated elliptical correlations can be negatively biased by as much as 70% of true values, thereby causing a large degree of estimation error.

Accordingly, our study focuses on improving the expectation vector and variance-covariance (VCV) matrix used as input to a range of MV optimization rules by using a model-based approach. Specifically we apply the Gaussian-copula-AR(2)-GARCH-GJR-Skew-T model that links asymmetric marginals with a symmetric copula. The Gaussian copula is a suitably parsimonious model for MV optimization applications. To allow for the autoregressive and asymmetric volatility clustering in the marginal returns distributions, we use the AR(2) and GARCH-GJR (Glosten et al., 1993) models, respectively. Skewness and kurtosis in the residuals are modeled using the Skewed Student t (Skew-T) of Hansen (1994). Our investigation is performed upon data sets consisting of international country portfolios, US industry indices and US Dow Jones Industrial Average (DJIA) stocks.

Our sample periods includes the volatile years from 2007-2009.

Our results show that across the 15 MV models investigated, superior and statistically significant improvements in terms of the Sharpe Ratio and CEQ are obtained when model-based estimates are used instead of historical sampling windows, even after accounting for realistic transaction costs. The models that exhibit the best performance are often variants of the minimum-variance (MIN) rule and the combination portfolios developed by Kan and Zhou (2007) and Tu and Zhou (2011). Based on additional analysis performed on the combination portfolio rules of Tu and Zhou (2011), we find that model-based estimates are more informative than historical samples, in the sense that the combination portfolios are improved due to an increased weight placed upon the optimization component of combination portfolio rather than the target component. Model-based estimates also result in improved performance compared to the base-case MV optimization model.

As implementation of portfolio optimization strategies bring higher re-balancing requirements, we find that outperforming the $1/N$ strategy in a statistically superior fashion after accounting for turnover and transaction costs remains an elusive challenge for MV optimization, even with the performance benefits emanating from applying model-based estimates. We find evidence of MV optimization rules incorporating model-based estimates outperforming the $1/N$ strategy only in the case of the US DJIA data set. DeMiguel et al. (2013b)'s use of option-implied volatility and skewness to improve estimates of expected return in MV optimization deliver similar findings - higher Sharpe Ratios are accompanied by higher turnover and transaction costs. Aside from very low turnover requirements, the $1/N$ produces superior performance when evaluated over long sample periods of several decades as it avoids concentrated positions, takes advantage of the mean-reversion effect when it sells high and buys low, captures size alpha as it overweights small-caps and underweights large-caps (Kritzman et al., 2010).

Based on our analysis, we also concur with Kritzman et al. (2010) that the poor performance of several MV optimization strategies as found by DeMiguel et al. (2009b) are a result of relying on raw historical returns in determining the expectation vector and VCV matrix. If an investor may only access estimates of expected returns based on historical samples, we find that the MIN variance strategy and the combination portfolio strategies of Tu and Zhou (2011) are the most robust to estimation error. A more sophisticated approach is that by applying estimates obtained from a model that is specified to account for asymmetries in the returns distribution, we find evidence that several MV optimization strategies are improved and thus MV continues to be a viable empirical framework for investors.

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Appendix A. Empirical design and model parametrization process

In the case of historical data, we follow the same approach as DeMiguel et al. (2009b) and Tu and Zhou (2011) where rolling sampling windows of historical returns are used to estimate the expected return vector and VCV matrix required as inputs into the MV optimization rules. More specifically, the process is as follows:

1. At time t, a rolling sampling window of size M months is selected.
2. During each month t, starting from $t = M + 1$, the returns data for the previous M months is used to calculate the one month ahead expected return ($\boldsymbol{\mu_t}$) and VCV matrix ($\boldsymbol{\Sigma_t}$).
3. Both the $\boldsymbol{\mu_t}$ and $\boldsymbol{\Sigma_t}$ are input to the various MV optimization rules as shown in Table 2.
4. The optimization rules produce the portfolio asset weights that are used to compute the portfolio return at month $t + 1$. Thus, a total of $T - M$ out-of-sample returns are produced by each of the models in Table 2, for each empirical data set shown in Table 1.
5. These out-of-sample returns and portfolio weights for each MV strategy are analyzed using a range of performance metrics and statistical measures that are reported in Subsection 4.

Our method is identical to the above approach, except that in Step 2, the rolling sampling windows of historical returns are first used to parameterize the Gaussian-copula-AR(2)-GARCH-GJR-Skew-T model,[15] and then the expected return ($\boldsymbol{\mu_t}$) and VCV matrix ($\boldsymbol{\Sigma_t}$) are estimated based on samples from this model. Specifically, Step 2 is replaced by the following process:

2a. During each month t, starting from $t = M + 1$, the returns data for the previous M months is used to fit the univariate AR(2)-GARCH-GJR process to the return series for a portfolio of N assets. For the residuals of each assets returns series, we adopt the Skew-T distribution of Hansen (1994).

2b. Based on the AR(2)-GARCH-GJR filtered return series of each asset, we apply the cumulative distribution function (CDF) function based on each univariate Skew-T distribution to obtain a set of uniform marginals. These N uniform marginals are used to calibrate the Gaussian copula to estimate the linear correlation matrix ($\mathbf{R}$).

2c. We generate $10,000$ uniformly distributed observations from the Gaussian copula for each asset, resulting in a total of $N \times 10,000$ observations for the entire portfolio. These simulated observations are translated back into the return series by applying the respective inverse CDF of each Skew-T marginal distribution, followed by their respective AR(2)-GARCH-GJR processes.

2d. The large simulated sample of portfolio returns is used to calculate the one month ahead expected mean ($\boldsymbol{\mu_t}$) and VCV matrix ($\boldsymbol{\Sigma_t}$).

[15] We parameterize the copula using the method of inference for margins (IFM). The IFM is a versatile two-step copula parametrization procedure that estimates the marginal distribution parameters and copula parameters separately. For more details, see Joe (1997).

Since we sample from a model that allows for asymmetries within the marginal distributions, we reduce the bias of the VCV matrix stemming from misspecified marginals.

Appendix B. Portfolio optimization frameworks

Appendix B.1. Benchmark and classic mean-variance (MV) models

The $1/N$ with rebalancing (EWR) approach creates a portfolio of N assets with a target weight of $1/N$ applied to each asset during each time period. Although the relative weights of each asset within the portfolio will change due to the fluctuation of returns during each period, the strategy continues to re-balance the weights equally each period. This strategy is recommended for use as a benchmark by DeMiguel et al. (2009b) to assess the performance of various portfolio rules due to its simplicity as it does not require the estimation of the moments of returns for input into the optimization rules. The $1/N$ without rebalancing (EWNR) benchmark variation distributes weights equally across the portfolio at the start and is left unadjusted for the rest of the investment horizon. This is similar to a buy-and-hold strategy where the investor exhibits inertia as the default asset allocation is accepted and avoids future re-investment and re-balancing decisions (and associated costs).

The in-sample mean-variance (MVIS) strategy uses mean-variance estimates that are based on the entire sample of asset returns and is an approximation of the true optimal rule that is otherwise unknown. An in-sample rule is unrealistic and not implementable in practice, however, it serves as a useful benchmark to measure how estimation errors affect the out-of-sample results of the different portfolio rules where historical returns or estimates of returns are used. In our study, the main benchmark is the EWR that we also refer to as the "$1/N$" strategy.

The sample-based mean-variance (MVS) strategy is the classic approach where historical mean returns and the VCV matrix are used to determine the weights for each out-of-sample period. No consideration is given within the optimization rule to adjust for estimation error in any form. Traditionally, many papers have used the MV model as a basic benchmark to compare their proposed model improvements.

Appendix B.1.1 Bayesian approach to estimation error

The Bayesian approach is one in which a predictive distribution of asset returns is used to estimate the mean return and VCV matrix. A certain subjective prior of $p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is selected and a returns distribution is generated by integration of the conditional likelihood $f(R|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ over that prior. Several variants are examined. The Bayesian diffuse-prior (BSD) is an approach that selects a prior to be diffuse with normal conditional likelihood, resulting in a predictive distribution that is a Student t with mean $\boldsymbol{\mu}$ and VCV matrix of $\boldsymbol{\Sigma}(1 + 1/M)$ (Barry, 1974; Klein and Bawa, 1976; Brown, 1979). Thus, expected returns are given by the historical mean and the sample VCV matrix increases in scale by $(1+1/M)$. M is the size of the sampling window. Similarly, the sample-based mean & adjusted VCV developed by Tu and Zhou (2011) (MVTZ), is an approach that scales the sample VCV matrix input as $\boldsymbol{\Sigma}(M/(M-N-2))$. These authors report that the resulting portfolio weights are unbiased and perform slightly better compared to the use of an unadjusted sample VCV matrix. However, it is mathematically evident that for both the BSD and MVTZ strategies, where long sampling windows are applied, scaling the VCV matrix will have negligible effects upon the resulting portfolio compared to an unscaled VCV matrix as applied in MVS.

The Bayes-Stein (BS) portfolio strategy (Stein, 1956; James and Stein, 1961) uses shrinkage estimators to manage errors in estimating expected returns and the VCV matrix. The intuition behind shrinkage estimators is that while shrinking an unbiased estimator towards a lower variance target has the advantage of reducing the variance of the estimator at the cost of introducing bias, it will perform well where the benefit of reduced variance outweighs the cost of the induced bias. Specific target means and the VCV matrix are selected and estimators are used to 'shrink' the sample mean and VCV matrix towards these values. Our implementation of the BS rule applies estimators for the target mean and VCV matrix as postulated by Jorion (1985, 1986). The target mean is selected to be the mean of the minimum variance portfolio. For the VCV matrix, a predictive variance of asset returns is calculated by using an informative prior on μ to calculate a precision value that calibrates the sample value of Σ accordingly (Jorion, 1986).

The Bayesian Data-and-Model (DM) strategy by Pástor (2000) follows similar principles to the BS approach except that the shrinkage targets and estimators are based on an asset pricing model instead. Specifically, the shrinkage target is based upon the Bayesian investor's prior belief in a chosen asset pricing model. The variance of the prior belief, relative to the information contained by the data, calibrates the amount of shrinkage estimation. Within the DM model, α is the Bayesian investor's prior that captures the extent of mispricing present within the asset pricing model. Similar to DeMiguel et al. (2009b), we assume α to follow a normal distribution where $\mu_\alpha = 0$ and $\sigma_\alpha = 1\%$ per annum. Intuitively, this translates to an investor believing that the mispricing of the asset pricing model, occurs with 95% probability within a 4% band around the estimated price on an annual basis. For the US Industry data sets, we implement the capital asset pricing model (CAPM), Fama and French (1993) 3-factor model and the Carhart (1997) 4-factor model and label these: 1-factor data-and-model (DM1), 3-factor data-and-model (DM3), and 4-factor data-and-model (DM4), respectively. Only the international capital asset pricing model (ICAPM) is used for the international country data set.

For the DM model, when historical returns samples are used as estimates of expected portfolio asset returns, the corresponding sampling windows for factor returns are used in the regression analysis component. However, when the Gaussian-copula-AR(2)-GARCH-GJR-Skew-T model is used to incorporate distributional asymmetries in the estimation of expected portfolio asset returns, the factor returns are generated by using univariate Monte-Carlo simulations. As our model-based estimation procedure generates $10,000$ returns for each asset in the portfolio, to perform the regression analysis required for the DM model, simulating 10,000 returns for each factor is necessary. The factor returns are modeled using the marginal models as described in Subsection 3.1.1 where they are assumed to be independent of one another (i.e., no dependence modeling) and parameterized upon sampling windows of the same size as applied for the portfolio asset returns.

Appendix B.1.2 Models based on moment restrictions

The MIN rule selects the portfolio of risky assets that minimizes the variance of portfolio returns. In its implementation, the expected returns are ignored entirely and only estimation of the VCV matrix is required. DeMiguel et al. (2009a,b) document that the best performing MV rules are often variants of the MIN. Ignoring expected returns successfully reduces the occurrence of extreme out-of-sample portfolio weights compared to the MVS (Jagannathan and Ma, 2003).

The MacKinlay and Pastor (2000) missing-factor model (MP) considers the case where if returns can be explained by a set of factors, it is possible that not all of these factors are observable and thus fail to be explicitly incorporated in the asset pricing model. As a result, any mispricing is contained within the VCV matrix of residuals. Based on this intuition, they design a more stable and reliable estimator of returns compared to the traditional MVS. For the implementation of this rule, we use the approach followed by Kan and Zhou (2007) and DeMiguel et al. (2009b) where an approximation function is used to determine the portfolio weights.

Appendix B.1.3 Models with portfolio constraints

The implementation of the sample-based mean-variance with shortsale constraints (MVC), Bayes-Stein with shortsale constraints (BSC) and minimum-variance with shortsale constraints (MINC) rules are the same as for their non-constrained counterparts except all weights are non-negative. Intuitively, all positions in the portfolio are long only. This approach accords with the fact that many funds in practice have short sales restrictions. Furthermore, empirical evidence shows that short-sales constrained portfolios usually exhibit better performance as they shrink the elements of the VCV matrix (Jagannathan and Ma, 2003).

The minimum-variance with generalized constraints (GMINC) rule is introduced by DeMiguel et al. (2009b) to further examine methods that account for correlations between returns, but continue to ignore expected returns to improve out of sample portfolio performance. The GMINC incorporates elements of the $1/N$ rule by constraining portfolio weights such that $w_t \geq \frac{1}{2}\frac{1}{N}\mathbf{1}_N$.

Appendix B.1.4 Combination rules

Combination rules can be considered shrinkage estimators that apply shrinkage upon portfolio weights (DeMiguel et al., 2009b; Tu and Zhou, 2011) but are different to the BS and DM portfolio rules that shrink towards target mean returns and VCV matrix. Combination rules are appealing since they allow the user to work directly with portfolio weights and shrink the portfolio towards specific, selected targets. Intuitively, combination rules consist of a target component and an optimization component. The optimization component usually exhibits a large degree of variance but is asymptotically unbiased. The target component is biased but usually exhibits little or no variance. Thus, combination portfolios are usually a tradeoff between bias and variance to reduce estimation error.

The combination portfolio (x_c) as shown in equation (B.1) is a weighted average combination of the optimization (x_α) and target (x_β) component rules. The coefficients (α, β) applied to each component rule are selected to optimally maximize the expected utility of the MV investor. Intuitively, if the sample data input into the combination portfolio strategy is a reliable, informative estimate, a higher weight will be applied to the optimization component. However, in instances where the true optimal rule is closer to the target component, the combination portfolio will assign a greater coefficient to the target component instead.

$$x_c = \alpha \cdot x_\alpha + \beta \cdot x_\beta, \text{ where } \mathbf{1}_N^\top x_c = 1, \tag{B.1}$$

The Kan and Zhou (2007) 'three-fund' model (MVMIN) is a combination of the MVS and MIN portfolio rule. The motivation behind Kan and Zhou (2007) is that due to the size of estimation errors that occur when holding the sample tangency portfolio and the risky-free asset, holding a third fund of risky assets such as the MIN portfolio diversifies away some of the estimation risk. Therefore, it is labeled the 'three-fund' model as it is a combination of the tangency portfolio on the MVS efficient frontier, the risk-free asset and the MIN portfolio. In this case, the optimization component is the MVS rule as it exhibits greater variance than the MIN rule.

The mixture of minimum-variance and $1/N$ (EWMIN) is a mixture of these component rules. DeMiguel et al. (2009b) proposes this rule that ignores estimates of expected returns, but continues to estimate the VCV matrix. Poor estimations of mean returns often lead to extreme out-of-sample portfolio weights, thus by ignoring them, DeMiguel et al. (2009b) intend to create an improved portfolio rule. In this scenario, the optimization component is the MIN rule, that exhibits higher variance than $1/N$ (zero variance).

Tu and Zhou (2011) propose four models that are combinations of the $1/N$ rule and other optimal MV rules, namely the original Markowitz rule and the extensions developed by Jorion (1986), MacKinlay and Pastor (2000) and Kan and Zhou (2007). These combination rules can be interpreted as shrinkage estimators where the $1/N$ strategy provides the target component. The degree of shrinkage between these two portfolios is chosen as an optimal tradeoff between the bias and variance as represented by the $1/N$ and optimal portfolios, respectively. We implement the combination of $1/N$ and Markowitz (1952) (EWMV) rule and combination of $1/N$ and Kan and Zhou (2007) (EWKZ) rule as they are the best performing of this group of models as shown in Tu and Zhou (2011) when empirical tests are performed upon historical data. Tu and Zhou (2011) find that the EWKZ is the best performing rule in their investigation. The EWKZ is a combination of the $1/N$ and MVMIN. As such, it is effectively a 'four-fund' portfolio rule as it is a combination of the risk-free asset, MVTZ, MIN, and the EWR portfolio rules. The optimization components of the EWMV and EWKZ are the MV and MVMIN rules, respectively.

Appendix C. Portfolio re-balancing analysis

Tokat and Wicas (2007) state that for practitioners, an optimal investment strategy involves three key issues: (1) frequency of re-balancing; (2) maximum threshold of deviations from target asset allocations allowed before triggering re-balancing; and (3) re-balancing fully towards the target allocation or an intermediate allocation. All three issues are related to reducing turnover and the impact of transaction costs (Pliska and Suzuki, 2004; Mendes and Marques, 2012). We focus on the final of these issues as it involves other important market frictions caused by low liquidity, compliance with regulatory restrictions and the market impact of trades. If one is unable to re-balance fully towards the target portfolio weights as required by the portfolio strategy, this results in suboptimal diversification. Therefore, other things equal, a strategy that leads to greater stability in target portfolio weights is desirable as it is easier for a practitioner to implement (DeMiguel and Nogales, 2009). As such, in addition to risk-adjusted performance, assessing the average standard deviation in target weights is a criteria that a practitioner uses in the selection of a portfolio strategy.

Accordingly, Table C.1 shows an analysis of the variability of target portfolio weights across each of our previous settings. The average standard deviation in target portfolio weights across the entire out-of-sample time period is calculated as shown in equation C.1:

$$\bar{\sigma}(\hat{w}_{k,c,M}) = \frac{\Sigma_{t=1}^{T-M}\sigma(\hat{w}_{k,t,c,M})}{T-M} \tag{C.1}$$

where

$$\sigma(\hat{w}_{k,t,c,M}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{w}_{k,t,c,M,i} - \bar{\hat{w}}_{k,t,c,M})} \tag{C.2}$$

where $\hat{w}_{k,t,c,M}$ is the N vector of target portfolio weights at time t under strategy k using estimates of expected returns from dataset c[16] based upon a window size of M. Similarly, $\hat{w}_{k,t,c,M,i}$ is the target portfolio weight for asset i in a portfolio of N assets, and $\bar{\hat{w}}_{k,t,c,M}$ is the average target portfolio weight across the portfolio of N assets.

Table C.1 shows the average standard deviation in target weights produced when the expected return vector and VCV matrix are based upon historical returns samples versus the model-based estimates, for sample windows of either 120 (Panel A) or 240 months (Panel B). Column Δ reports the difference between the average standard deviation in target portfolios weights of the two approaches. A positive value given by Δ indicates an improvement when model-based estimates are

[16] In our implementation, c is either the historical returns samples or model-based estimates.

Table C.1: Standard deviation of target weights across alternative optimization rules

This table shows the average standard deviation in target weights when historical samples or model-based estimates are used for each of the three data sets investigated. See Table 2 for definitions of the rules. The Δ column indicates the difference between the standard deviation of weights: historical minus model-based - with a positive value indicating an improvement (i.e., reduction) for the model-based approach. Panel A and Panel B show the performance of the portfolio rules for sample window lengths of 120 and 240 months, respectively.

	International Country Indices			US Industry Indices			US DJIA Stocks		
Portfolio strategy	Hist. samples	Model based est.	Δ	Hist. samples	Model based est.	Δ	Hist. samples	Model based est.	Δ
Panel A: Sample window length M = 120									
Classic approach that ignores estimation error									
MVS	2.60	1.45#	1.16*	2.73	1.31#	1.42*	0.19	0.12#	0.07*
Bayesian approach to estimation error									
BSD	2.60	1.45#	1.16*	2.73	1.31#	1.42*	0.19	0.12#	0.07*
MVTZ	2.60	1.45#	1.16*	2.73	1.31#	1.42*	0.19	0.12#	0.07*
BS	1.35	0.60#	0.75*	1.05	0.42#	0.63*	0.13	0.13	0.00
DM1	0.19	1.54	-1.36*	0.09	1.28	-1.19*	0.09	0.08#	0.01
DM3	n.a.	n.a.	n.a.	0.09	1.21	-1.13*	0.19	0.13#	0.06*
DM4	n.a.	n.a.	n.a.	0.08	1.02	-0.94*	0.19	0.14#	0.06*
Moment restrictions									
MP	0.02	0.09	-0.07*	0.01	0.05#	-0.04*	0.07	0.07	0.00
MIN	0.23	0.24#	-0.01	0.26[a,b]	0.16#	0.10*	0.08	0.11#	-0.03
Portfolio constraints									
MVC	0.28	0.26#	0.02	0.22[b]	0.21	0.01	0.15	0.17	-0.02
BSC	0.22	0.23#	-0.01	0.18	0.17	0.01	0.14	0.17#	-0.03*
MINC	0.16	0.19#	-0.03*	0.13	0.12#	0.01	0.03	0.02	-0.06*
GMINC	0.12[b]	0.12#	0.00	0.09[a,b]	0.09#	0.00	0.01	0.01	0.00
Combination portfolios									
MVMIN	0.99	0.52#	0.47*	0.45	0.21#	0.24*	0.10	0.12#	-0.02
EWMIN	0.13	0.21	-0.08*	0.18	0.12#	0.06*	0.04	0.10#	-0.06*
EWKZ	0.39	1.25	-0.85*	0.41	0.17#	0.24*	0.10	0.07#	0.03*
EWMV	0.67	0.29#	0.38*	0.30	0.13#	0.17*	0.20	0.15#	0.05*
Panel B: Sample window length M = 240									
Classic approach that ignores estimation error									
MVS	0.45	0.29#	0.16*	0.88	0.34#	0.54*	0.13	0.09#	0.04*
Bayesian approach to estimation error									
BSD	0.45	0.29#	0.16*	0.88	0.34#	0.54*	0.13	0.09#	0.04*
MVTZ	0.45	0.29#	0.16*	0.88	0.34#	0.54*	0.13	0.09#	0.04*
BS	0.25	0.21#	0.04*	0.39	0.46#	-0.08*	0.09	0.10	-0.01
DM1	0.17	0.29	-0.12*	0.07	0.34	-0.27*	0.07	0.09#	-0.02
DM3	n.a.	n.a.	n.a	0.07	0.36	-0.29*	0.13	0.09#	0.04*
DM4	n.a.	n.a.	n.a	0.06	0.36	-0.30*	0.13	0.09#	0.04*
Moment restrictions									
MP	0.02	0.03	-0.01	0.01	0.02#	-0.01	0.11	0.02	0.01
MIN	0.21	0.18#	0.03*	0.23	0.14#	0.09*	0.07	0.09#	-0.02
Portfolio constraints									
MVC	0.24	0.25#	0.00	0.20	0.19#	0.01	0.13	0.18	-0.04*
BSC	0.18	0.18#	0.00	0.17	0.15#	0.02	0.12	0.18	-0.05*
MINC	0.18	0.16#	0.02	0.14[b]	0.12	0.02	0.03	0.01#	0.02
GMINC	0.13[b]	0.11#	0.02	0.10[b]	0.09	0.01	0.02	0.01#	0.01
Combination portfolios									
MVMIN	0.22	0.18#	0.04*	0.24	0.14#	0.10*	0.07	0.09#	-0.02
EWMIN	0.14	0.14#	0.00	0.19	0.11#	0.07*	0.05	0.07#	-0.02
EWKZ	0.17	0.15#	0.02	0.16	0.11#	0.06*	0.07	0.06#	0.01
EWMV	0.16	0.13#	0.03*	0.23	0.07#	0.16*	0.13	0.09#	0.04*

# indicates a higher CEQ value when model-based estimates are applied compared to historical samples.

* indicate that the values in Δ are significantly different from zero using the two-sided, non-parametric Wilcoxon rank-sum test at the 5% level.

applied, compared to historical returns samples. We test the statistical significance of the values of Δ by using the two-sided, non-parametric Wilcoxon rank-sum test at the 5% level (Wilcoxon, 1945; Siegel, 1956). To assess whether the use of model-based estimates are able to simultaneously improve portfolio performance and reduce the average standard deviation of portfolio weights, we identify cases where the portfolio strategy produces superior portfolio performance than historical returns samples ("#"). The portfolio performance is evaluated using the CEQ measure.

In Table C.1, Panels A and B across all three data sets show that the MVS, BSD, and MVTZ portfolios exhibit a significant reduction in the average standard deviation of portfolio target weights when model-based estimates are used. These portfolios also exhibit substantial improvement in terms of their CEQ for the model-based approach. Strategies within the portfolio constraints category (MVC, BSC, MINC, GMINC) exhibit the least improvement and intuitively, this is logical as the portfolio weights are already constrained to vary within a narrow band of values. Therefore, there is a low likelihood of model-based estimates being able to reduce the average standard deviation of target portfolio weights even further.

For the international country portfolio analysis in Panel A, we find that 6 portfolio rules investigated show a reduction in the standard deviation of target portfolio weights that are significant at the 5% level. However, in Panel B, for the same data set with a longer sampling window, while we find that 7 portfolio rules are improved, the reduction in average standard deviations of target portfolio weights are much smaller. This is a similar effect to that found in previous sections where estimates parameterized on longer sampling windows often enhances a larger number of portfolio strategies but the degree of improvement decreases. Intuitively, with longer sampling windows, the model-based estimates have less error to reduce. Thus, shorter windows that exhibit greater error result in greater improvement when the model-based estimates that account for asymmetries are accounted for.

For the US industry setting (17 indices), nine (eight) of the portfolio rules in Panel A (Panel B) exhibit significantly lower average standard deviations in target portfolio weights when model-based estimates are used as opposed to those based on historical samples. The DM portfolio strategies exhibit increased average standard deviations in portfolio weights whereas the differences in the MP strategy and strategies within the portfolio constraint category, are insignificant at the 5% level. In the US DJIA dataset (30 stocks), the reduction in the standard deviation of portfolio weights are lower compared to the previous, smaller portfolios as the portfolio weights are less concentrated and spread out across a larger number of assets. For this case, Panel A (Panel B), shows that seven (six) strategies exhibit statistically significant and reduced average standard deviations when model-based estimates are applied. For both US data sets, similar to the international country data set, we find that when the sampling windows are lengthened, the reduction in the average standard deviation in portfolio weights also decreases.

Intuitively, portfolio managers adjust asset weights to track a benchmark index or investor's optimal utility as closely as possible (Pliska and Suzuki, 2004). Therefore, poor estimates of expected returns can lead to extreme adjustments of portfolio weights that result in severely 'over-shooting' or 'under-shooting' the investor's optimal utility. These large degrees of tracking error are undesirable even though they can lead to lower turnover. Alternatively, a more reliable estimate of expected returns should result in improved performance and lead to consistent, smaller adjustments to target portfolio weights to reduce the likelihood of 'over-shooting' or 'under-shooting'. This is analogous to the process of 'fine-tuning' that leads to smaller errors in tracking the investor's optimal utility at the cost of increased turnover.

Thus, the Gaussian-copula-AR(2)-GARCH-GJR-Skew-T model provides enhanced estimates of the mean vector and VCV matrix that lead to improved performance with more frequent but smaller adjustments to portfolio weights to track the optimal target MV portfolio. From the results in Tables 3, 4, and 5, even after accounting for transaction costs, the application of the model-based estimates continues to produce superior and statistically different Sharpe Ratios and CEQ values compared to use of historical returns. In Table C.1, we find that most of the portfolio strategies that demonstrate superior performance outcomes with the application of the model-based estimates, also exhibit the desirable attribute of reduced average standard deviation in target portfolio weights in comparison to historical samples.

Research Highlights

- Model-based estimates that incorporate return asymmetries are applied to 18 mean-variance optimization rules.
- Model-based estimates are a significant improvement over use of historical-based estimates.
- Model-based estimates result in out-performance of the basic mean-variance optimization strategy after transaction costs.
- Outperforming the 1/N portfolio after transaction costs remains an elusive task even with model-based estimates.