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ASYMPTOTICS OF BOND YIELDS AND VOLATILITIES FOR EXTENDED VASICEK MODELS UNDER THE REAL-WORLD MEASURE

K. FERGUSSON

ABSTRACT. Vasicek's short rate model is a mean reverting model of the short rate which permits closed-form pricing formulae of zero coupon bonds and options on zero coupon bonds. This article supplies proofs which are valid for any single factor mean reverting Gaussian short rate model having time-inhomogeneous parameters. The formulae are for the expected present value of payoffs under the real-world probability measure, known as actuarial pricing. Importantly, we give formulae for asymptotic levels of bond yields and volatilities for extended Vasicek models when suitable conditions are imposed on the model parameters.

1. INTRODUCTION

In actuarial science the short term interest rate plays a central role in valuations of future cashflows, particularly those pertaining to short-tail insurance policies. A short rate model is a mathematical model of the instantaneous, continuously compounded deposit rate for a specific currency. The most realistic proxy for the short rate among investible securities is probably the overnight cash deposit rate, expressed as a continuously compounded rate. Short rates are typically modelled as stochastic processes and coverages of short rate models can be found, for example, in Rebonato [1998] and Brigo and Mercurio [2006].

The short rate models considered in this paper are called Gaussian short rate models and are specified by stochastic differential equations (SDEs) with a single noise source and with time-dependent coefficients. From an actuarial pricing perspective the availability of explicit pricing formulae for calculations involving the short term interest rate is of extreme importance. The class of short rate dynamics where one can probably expect the widest range of explicit valuation formulae is probably the Gaussian class. They are convenient and also reasonably realistic for pricing future cash flows and contingent claims. They have explicit closed-form formulae for their transition density functions and also allow negative values. Of particular importance for actuaries is the requirement that the long-term bond yield implied by the model be a finite constant, which is guaranteed for the Vasicek model

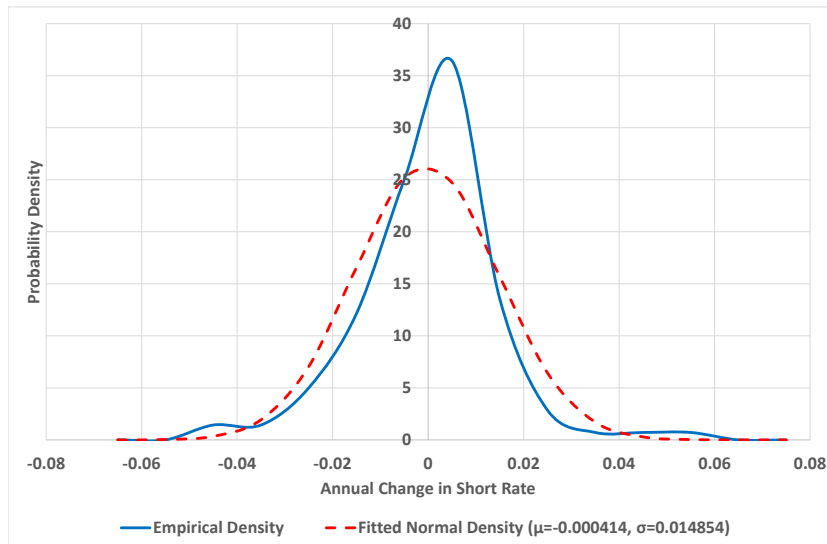
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FIGURE 1. Comparison of empirical probability density function of annual change in short rate with that of the fitted normal distribution (US 1Y cash rates 1871 - 2010).



but not necessarily for extended Vasicek models. Figure 1 illustrates the asymmetry of the distribution of annual changes in the short rate for US cash rates, which corresponds to the leverage effect in bond markets. While this effect is not captured by extended Vasicek models, it is a short-term effect which is less pronounced when analysing the asymptotic behaviour of bond yields and volatilities.

A particular example of a Gaussian short rate model is the well-known Vasicek model, which is a linear mean reverting stochastic model, see Vasicek [1977]. This ensures that interest rates adhere to a long run reference level.

Working on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, P)$, the SDE for the extended Vasicek short rate model or Hull-White extension is given as

$$(1.1) \quad dr_t = \kappa_t(\bar{r}_t - r_t)dt + \sigma_t dZ_t,$$

where r_t is the short rate at time $t \geq 0$, Z_t is a Wiener process adapted to the filtration $(\mathcal{A}_t)_{t \geq 0}$ and \bar{r} , κ and σ are positive deterministic functions of time. The aim of this paper is to provide for this type of model a wide range of valuation formulae that are useful in actuarial valuations and, for suitable conditions on the model parameters, to show that the implied long-term bond yield is finite.

In Section 2 we introduce actuarial pricing and in Section 3 we describe the Vasicek short rate model and the Hull-White extension, the explicit solutions of which are given in Section 4. We provide formulae for the transition density function in Sections 5 and 6, the zero coupon bond (ZCB) price in Section 7, bond yields and instantaneous forward rates in Section 8 and the price of an option on the ZCB

in Section 9. We go deliberately through all steps of the derivations, even though some of these may be well-known under risk neutral assumptions. Note, however, our pricing will be done under the real-world probability measure P and not under some assumed risk neutral measure.

2. ACTUARIAL PRICING

Let

$$(2.1) \quad B_t = B_0 \exp \left\{ \int_0^t r_s ds \right\}$$

denote for $t \geq 0$ the savings account, the locally risk-free asset. Actuarial pricing, at the present time t , of a contingent claim H_T occurring at some future time T is computed using the actuarial pricing formula

$$(2.2) \quad V_t = E \left(\frac{B_t}{B_T} H_T \middle| \mathcal{A}_t \right),$$

where $E(\cdot | \mathcal{A}_t)$ denotes conditional expectation under the real-world probability measure P . It is the expected present value of the contingent claim H_T occurring at time T . Under the classical risk neutral assumptions every discounted price process, discounted by the savings account, is forming a martingale under some risk neutral probability measure Q . The above actuarial pricing formula arises from the risk neutral pricing formula

$$(2.3) \quad V_t^* = E^Q \left(\frac{B_t}{B_T} H_T \middle| \mathcal{A}_t \right),$$

when the random variable

$$(2.4) \quad \frac{B_t}{B_T} H_T$$

is independent from the Radon-Nikodym derivative $\frac{dQ}{dP} |_{\mathcal{A}_T}$, where Q is an assumed risk neutral measure, equivalent to the real-world probability measure P . For most models considered in the literature, such an equivalent risk neutral measure is assumed to exist, but for more general models this may not be the case.

To cover such cases, pricing under the real-world probability measure has been formalised with Platen's benchmark approach, described in Platen [2002] and Platen and Heath [2006], and the above actuarial pricing formula emerges also as a special case under this approach. In particular, the benchmark approach allows one to apply formally risk neutral pricing even when an equivalent risk neutral probability measure does not exist, see Platen and Taylor [2016]. Therefore, the pricing formulae developed in this paper can be applied more widely than typically believed.

From the actuarial pricing formula (2.2) it follows in the case when the contingent claim is independent from the short rate that one has

$$(2.5) \quad V_t = E \left(\frac{B_t}{B_T} \middle| \mathcal{A}_t \right) E(H_T | \mathcal{A}_t) = G_T(t) E(H_T | \mathcal{A}_t)$$

for $t \in [0, T]$. Here $G_T(t)$ is the price of a ZCB maturing at time T , and we have the actuarial price $G_T(t)$ at time $t \in [0, T]$ as

$$(2.6) \quad G_T(t) = E \left(\exp \left\{ - \int_t^T r_s ds \right\} \middle| \mathcal{A}_t \right).$$

Thus the price of a ZCB plays a central role in actuarial pricing.

For a \bar{T} -expiry call option, $0 < \bar{T} \leq T < \infty$, on such a ZCB with strike price K , the actuarial price $c_{\bar{T},T,K}(t)$ at time $t \in [0, T]$ is

$$(2.7) \quad c_{\bar{T},T,K}(t) = E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} (G_T(\bar{T}) - K)^+ \middle| \mathcal{A}_t \right).$$

Corresponding formulae for a put option, asset-or-nothing call option, asset-or-nothing put option, cash-or-nothing call option and cash-or-nothing put option, each on such a ZCB, are

$$(2.8) \quad \begin{aligned} p_{\bar{T},T,K}(t) &= E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} (K - G_T(\bar{T}))^+ \middle| \mathcal{A}_t \right) \\ A_{\bar{T},T,K}^+(t) &= E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} G_T(\bar{T}) \mathbf{1}_{G_T(\bar{T}) > K} \middle| \mathcal{A}_t \right) \\ A_{\bar{T},T,K}^-(t) &= E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} G_T(\bar{T}) \mathbf{1}_{G_T(\bar{T}) \leq K} \middle| \mathcal{A}_t \right) \\ B_{\bar{T},T,K}^+(t) &= E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} \mathbf{1}_{G_T(\bar{T}) > K} \middle| \mathcal{A}_t \right) \\ B_{\bar{T},T,K}^-(t) &= E \left(\exp \left\{ - \int_t^{\bar{T}} r_s ds \right\} \mathbf{1}_{G_T(\bar{T}) \leq K} \middle| \mathcal{A}_t \right), \end{aligned}$$

respectively. Here $\mathbf{1}_{X > K}$ denotes the indicator function, equalling one if the random variable X exceeds the value K , and zero otherwise.

The availability of explicit formulae for the transition density function of the short rate makes it not only possible to provide explicit formulae for the above prices but also to fit the model to historical data using maximum likelihood estimation, as demonstrated in Fergusson and Platen [2015]. In determining the formulae for ZCBs and ZCB options we derive also a formula for the moment generating function of $\log(B_{\bar{T}}/B_t)$, which can be used for the approximation of other prices.

3. VASICEK SHORT RATE MODEL AND EXTENSIONS

The Vasicek model was proposed in Vasicek [1977], and extended in Hull and White [1990] to the Hull-White model whose drift and diffusion parameters are made time dependent, which also became known as the extended Vasicek model.

This SDE (1.1) is the Ornstein-Uhlenbeck SDE whose explicit solution is obtained by solving the SDE of $q_t = r_t e_t$ with

$$(3.1) \quad e_t = \exp \left\{ \int_0^t \kappa_s ds \right\},$$

where

$$(3.2) \quad dq_t = d(r_t e_t) = \kappa_t e_t \bar{r}_t dt + e_t \sigma_t dZ_t.$$

Vasicek's model, which is a special case of (1.1) with κ_t , \bar{r}_t , σ_t constant, and whose SDE is

$$(3.3) \quad dr_t = \kappa(\bar{r} - r_t) dt + \sigma dZ_t,$$

was probably the first interest rate model to capture mean reversion, an essential characteristic of the interest rate that sets it apart from simpler models. Thus,

under the real-world probability measure, as opposed to stock prices, for instance, interest rates are not expected to rise indefinitely. This is because at very high levels they would hamper economic activity, prompting a decrease in interest rates. Similarly, interest rates are unlikely to decrease indefinitely. As a result, interest rates move mainly in a range, showing a tendency to revert to a long run value.

The drift factor $\kappa(\bar{r} - r_t)$ represents the expected instantaneous change in the interest rate at time t . The parameter \bar{r} represents the long run reference value towards which the interest rate reverts. Indeed, in the absence of uncertainty, the interest rate would remain constant when it has reached $r_t = \bar{r}$. The parameter κ , governing the speed of adjustment, needs to be positive to ensure stability around the long term value. For example, when r_t is below \bar{r} , the drift term $\kappa(\bar{r} - r_t)$ becomes positive for positive κ , generating a tendency for the interest rate to move upwards.

The main disadvantage seemed that, under Vasicek's model, it is theoretically possible for the interest rate to become negative. In the previous academic literature this has been interpreted as an undesirable feature. However, on several occasions the market generated in recent years negative interest rates, for example in Switzerland and in Europe. The possibility of negative interest rates is excluded in the Cox-Ingersoll-Ross model (see Cox et al. [1985]), the exponential Vasicek model (see Brigo and Mercurio [2001]), the model of Black et al. [1990] and the model of Black and Karasinski [1991], among many others. See Brigo and Mercurio [2006] for further discussions.

Another disadvantage is that the Vasicek model does not capture stochastic volatility, evident in the graph of the quadratic variation of the short rate in Figure 2. Therefore, a serious consideration of real-world dynamics would require models whose stochastic differential equations of the short rate have stochastic volatility, such as the Cox-Ingersoll-Ross model and the 3/2 model (see Platen [1999]). However, owing to the mean reverting nature of stochastic volatility, this will have less impact on the asymptotic behaviour of bond volatilities.

The Vasicek model was further extended in the Hull-White model (see Hull and White [1990]), by allowing time dependence in the drift parameters. The Hull-White model is specified by the SDE

$$(3.4) \quad dr_t = \{\theta(t) + a(t)(b - r_t)\}dt + \sigma(t)dZ_t,$$

where $\theta(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of t , satisfying $a(t) > 0$ and $\sigma(t) > 0$ and b is a constant. When setting $\kappa_t = a(t)$ and $\bar{r}_t = b + \theta(t)/a(t)$ in (1.1) we obtain (3.4). Further, in (3.4) when setting $a(t) = 0$ and $\sigma(t)$ equal to a positive constant σ we obtain

$$(3.5) \quad dr_t = \theta(t) dt + \sigma dZ_t,$$

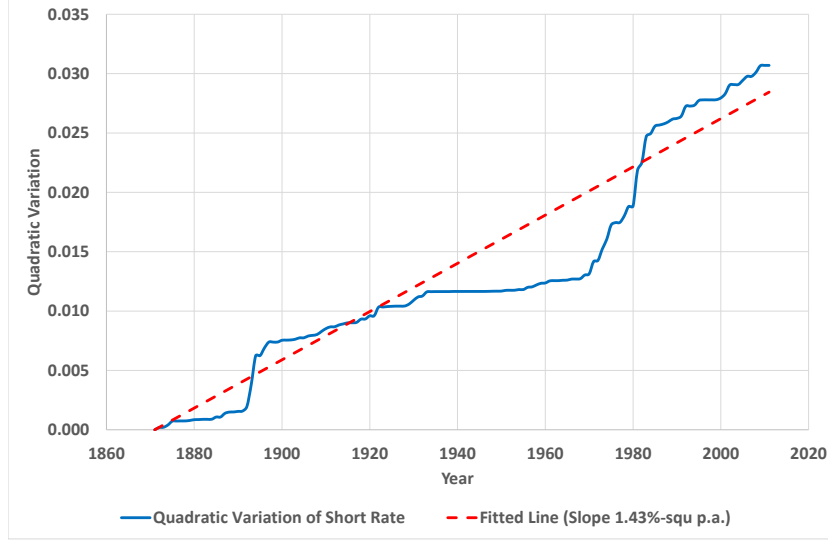
which is implicitly what is employed in Ho and Lee [1986].

We now provide an explicit solution to each of the SDE (3.3) and the SDE (3.4) from which we determine the associated transition density function.

4. EXPLICIT FORMULA FOR THE SHORT RATE

An explicit solution to the Ornstein-Uhlenbeck process is straightforwardly obtained in the following theorem.

FIGURE 2. Quadratic variation of the short rate (US 1Y cash rates 1871 - 2010).



Proposition 1. *The short rate r_t satisfying the Vasicek SDE (3.3) has solution*

$$(4.1) \quad r_t = r_s \exp(-\kappa(t-s)) + \bar{r}(1 - \exp(-\kappa(t-s))) + \sigma \int_s^t \exp(-\kappa(t-u)) dZ_u$$

for times s and t with $0 \leq s < t$ and for positive constants \bar{r} , κ and σ . Here Z is the Wiener process in (3.3).

Proof. Integrating both sides of (3.2) between times s and t gives

$$(4.2) \quad r_t \exp(\kappa t) - r_s \exp(\kappa s) = \kappa \bar{r} \int_s^t \exp(\kappa u) du + \sigma \int_s^t \exp(\kappa u) dZ_u.$$

Multiplying both sides by $\exp(-\kappa t)$ and simplifying gives (4.1). \square

The proof is similar for the solution to the Hull-White SDE in (3.4).

Proposition 2. *The short rate r_t satisfying the Hull-White SDE (3.4) has solution*

$$(4.3) \quad r_t = r_s \exp \left\{ - \int_s^t a(\tau) d\tau \right\} + \int_s^t \exp \left\{ - \int_u^t a(\tau) d\tau \right\} \{ \theta(u) + a(u)b \} du \\ + \int_s^t \exp \left\{ - \int_u^t a(\tau) d\tau \right\} \sigma(u) dZ_u$$

for times s and t with $0 \leq s < t$, for positive functions $\theta(u)$, $a(u)$ and $\sigma(u)$ and for a constant b . Here Z is the Wiener process in (3.4).

5. TRANSITION DENSITY OF THE SHORT RATE

As is the case for the Ho-Lee model in (3.5) and the Hull-White model in (3.4), the transition density function of the Vasicek short rate is that of a normal distribution.

Corollary 1. *For times s and t with $0 \leq s < t \leq T$ the transition density of the short rate r_t in (3.3) is given by*

(5.1)

$$p_r(s, r_s, t, r_t) = \frac{1}{\sqrt{2\pi\sigma^2 \frac{1 - \exp(-2\kappa(t-s))}{2\kappa}}} \times \exp\left(-\frac{1}{2} \left(\frac{r_t - r_s \exp(-\kappa(t-s)) - \bar{r}(1 - \exp(-\kappa(t-s)))}{\sqrt{\sigma^2 \frac{1 - \exp(-2\kappa(t-s))}{2\kappa}}}\right)^2\right).$$

Proof. From (4.1) we see that r_t conditioned upon r_s is normally distributed and has expected value

$$(5.2) \quad E(r_t | \mathcal{A}_s) = r_s \exp(-\kappa(t-s)) + \bar{r}(1 - \exp(-\kappa(t-s)))$$

and variance

$$(5.3) \quad VAR(r_t | \mathcal{A}_s) = \sigma^2 \int_s^t \exp(-2\kappa(t-u)) du = \sigma^2(1 - \exp(-2\kappa(t-s)))/(2\kappa).$$

The transition density function must, therefore, be given by (5.1). \square

A graph of the transition density function is shown in Figure 3 for parameters shown in (5.4). These are maximum likelihood estimates derived from Shiller's annual data set of one-year US deposit rates from 1871 to 2012, with standard errors shown in brackets.

$$(5.4) \quad \begin{aligned} \bar{r} &= 0.042994 (0.0080023) \\ \kappa &= 0.162953 (0.053703) \\ \sigma &= 0.015384 (0.00099592). \end{aligned}$$

As for other Gaussian short rate models such as the Ho-Lee model and the Hull-White model, a potential disadvantage of the Vasicek model is the possibility of negative interest rates.

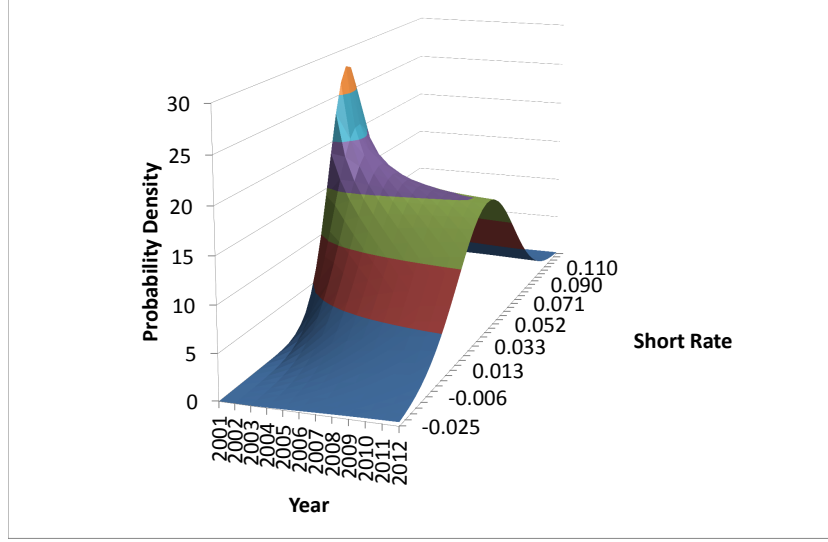
We show the parameter estimate for the mean reverting level \bar{r} alongside the historical short rates in Figure 4. We note that for the periods after 1930 a time dependent reference level may be appropriate but we deliberately keep constant parameters in this paper to clarify the methodology.

A lemma, which can be deduced from Corollary 1 and which will be used later, is as follows.

Lemma 1. *For the Vasicek process in (3.3) and times s, t with $s \leq t$ let the mean and variance of r_t given r_s be defined as*

$$(5.5) \quad \begin{aligned} m_s(t) &= E(r_t | \mathcal{A}_s) \\ v_s(t) &= VAR(r_t | \mathcal{A}_s) = E((r_t - m_s(t))^2 | \mathcal{A}_s). \end{aligned}$$

FIGURE 3. Vasicek transition density function of US cash rates based at year 2000 and short rate 0.064.



Then we have the explicit formulae

$$(5.6) \quad \begin{aligned} m_s(t) &= \bar{r}\kappa B(s, t) + r_s(1 - \kappa B(s, t)) \\ v_s(t) &= \sigma^2 \left(B(s, t) - \frac{1}{2}\kappa B(s, t)^2 \right), \end{aligned}$$

where

$$(5.7) \quad B(s, t) = (1 - \exp(-\kappa(t - s)))/\kappa.$$

Proof. Integrating the SDE (3.3) gives

$$(5.8) \quad r_t = r_s + \int_s^t \kappa(\bar{r} - r_u)du + \int_s^t \sigma dZ_u$$

and taking expectations conditioned on r_s gives

$$(5.9) \quad m_s(t) = r_s + \int_s^t \kappa(\bar{r} - m_s(u))du.$$

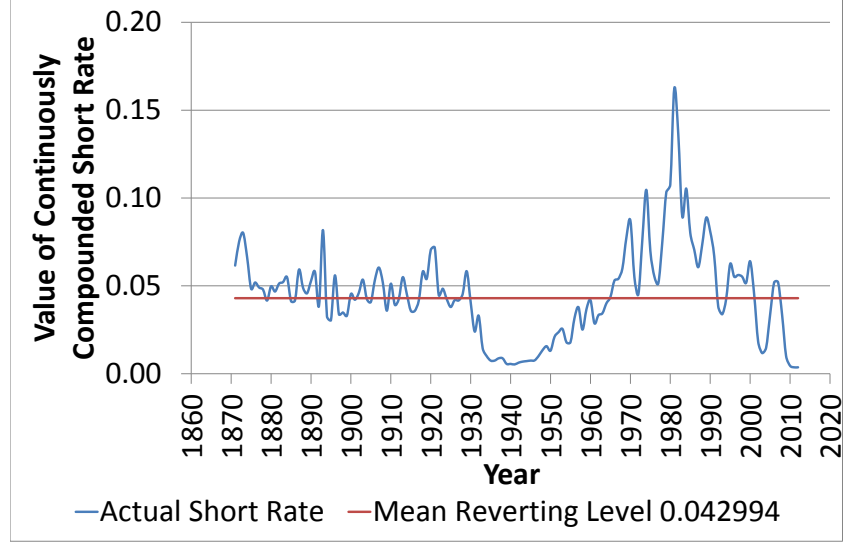
This can be written as a first order ordinary differential equation in $m_s(t)$

$$(5.10) \quad m_s(t)' = \kappa(\bar{r} - m_s(t))$$

with initial condition $m_s(s) = r_s$, the solution of which is straightforward. Now the SDE of r_t^2 is, by Ito's Lemma,

$$(5.11) \quad dr_t^2 = (\sigma^2 + 2\kappa\bar{r}r_t - 2\kappa r_t^2)dt + 2\sigma r_t dZ_t$$

FIGURE 4. Actual short rate and fitted Vasicek mean reverting level for US cash rates.



and integrating this SDE gives

$$(5.12) \quad r_t^2 = r_s^2 + \int_s^t (\sigma^2 + 2\kappa\bar{r}r_u - 2\kappa r_u^2) du + \int_s^t 2\sigma r_u dZ_u.$$

Taking expectations conditioned on r_s , and defining $m_s^{(2)}(t) = E(r_t^2 | \mathcal{A}_s)$, gives

$$(5.13) \quad m_s^{(2)}(s) = r_s^2 + \int_s^t (\sigma^2 + 2\kappa\bar{r}m_s(u) - 2\kappa m_s^{(2)}(u)) du$$

from which we have the ordinary differential equation

$$(5.14) \quad m_s^{(2)'}(t) = \sigma^2 + 2\kappa\bar{r}m_s(t) - 2\kappa m_s^{(2)}(t).$$

Multiplying both sides by $\exp(2\kappa t)$ and rearranging gives

$$(5.15) \quad \frac{d}{dt} \left(\exp(2\kappa t) m_t^{(2)} \right) = \sigma^2 \exp(2\kappa t) + 2\kappa\bar{r}m_s(t) \exp(2\kappa t)$$

$$(5.16) \quad = \sigma^2 \exp(2\kappa t) + 2\kappa\bar{r} \exp(2\kappa t) \left(\kappa\bar{r}B(s, t) + r_s(1 - \kappa B(s, t)) \right)$$

$$= (\sigma^2 + 2\kappa\bar{r}r_s) \exp(2\kappa t) + 2\kappa^2\bar{r}(\bar{r} - r_s) \exp(2\kappa t) B(s, t).$$

We note that

$$(5.17) \quad \int_s^t \exp(2\kappa u) B(s, u) du = \frac{1}{2} \exp(2\kappa t) B(s, t)^2$$

$$\int_s^t \exp(2\kappa u) du = \frac{1}{2} \exp(2\kappa t) (2B(s, t) - \kappa B(s, t)^2).$$

Therefore, integrating both sides of (5.15) from s to t gives

$$(5.18) \quad \begin{aligned} \exp(2\kappa t) m_s^{(2)}(t) &= \exp(2\kappa s) r_t^2 + (\sigma^2 + 2\kappa \bar{r} r_s) \frac{1}{2} \exp(2\kappa t) (2B(s, t) - \kappa B(s, t)^2) \\ &\quad + 2\kappa^2 \bar{r} (\bar{r} - r_s) \frac{1}{2} \exp(2\kappa t) B(s, t)^2 \\ &= \exp(2\kappa s) r_t^2 + (\sigma^2 + 2\kappa \bar{r} r_s) \exp(2\kappa t) B(s, t) \\ &\quad + \frac{1}{2} \exp(2\kappa t) B(s, t)^2 \left(2\kappa^2 \bar{r} (\bar{r} - r_s) - \kappa (\sigma^2 + 2\kappa \bar{r} r_s) \right) \\ &= \exp(2\kappa s) r_t^2 + (\sigma^2 + 2\kappa \bar{r} r_s) \exp(2\kappa t) B(s, t) \\ &\quad + \frac{1}{2} \exp(2\kappa t) B(s, t)^2 \left(2\kappa^2 \bar{r}^2 - \kappa \sigma^2 - 4\kappa^2 \bar{r} r_s \right). \end{aligned}$$

and dividing both sides by $\exp(2\kappa t)$ gives

$$(5.19) \quad \begin{aligned} m_s^{(2)}(t) &= r_s^2 \exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa \bar{r} r_s) B(s, t) \\ &\quad + \frac{1}{2} B(s, t)^2 \left(2\kappa^2 \bar{r}^2 - \kappa \sigma^2 - 4\kappa^2 \bar{r} r_s \right). \end{aligned}$$

The variance is computed as $v_s(t) = m_s^{(2)}(t) - (m_s(t))^2$, that is

$$(5.20) \quad \begin{aligned} v_s(t) &= r_s^2 \exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa \bar{r} r_s) B(s, t) \\ &\quad + \frac{1}{2} B(s, t)^2 \left(2\kappa^2 \bar{r}^2 - \kappa \sigma^2 - 4\kappa^2 \bar{r} r_s \right) \\ &\quad - \left(\bar{r} \kappa B(s, t) + r_s (1 - \kappa B(s, t)) \right)^2 \\ &= r_s^2 \exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa \bar{r} r_s) B(s, t) \\ &\quad + \frac{1}{2} B(s, t)^2 \left(2\kappa^2 \bar{r}^2 - \kappa \sigma^2 - 4\kappa^2 \bar{r} r_s \right) \\ &\quad - \bar{r}^2 \kappa^2 B(s, t)^2 - r_s^2 (1 - \kappa B(s, t))^2 - 2\kappa \bar{r} r_s (B(s, t) - \kappa B(s, t)^2) \\ &= \sigma^2 B(s, t) - \frac{1}{2} \kappa \sigma^2 B(s, t)^2 \end{aligned}$$

as required. \square

For the Hull-White model we have the following corollary.

Corollary 2. For times s and t with $0 \leq s < t \leq T$ the transition density of the short rate r_t in (3.4) is given by

$$(5.21) \quad p_r(s, r_s, t, r_t) = \frac{1}{\sqrt{2\pi v_s(t)}} \times \exp\left(-\frac{1}{2} \left(\frac{r_t - m_s(t)}{\sqrt{v_s(t)}}\right)^2\right),$$

where

$$(5.22) \quad \begin{aligned} m_s(t) &= r_s \exp\left\{-\int_s^t a(\tau) d\tau\right\} + \int_s^t \exp\left\{-\int_u^t a(\tau) d\tau\right\} \{\theta(u) + a(u)b\} du \\ v_s(t) &= \int_s^t \exp\left\{-2\int_u^t a(\tau) d\tau\right\} \sigma(u)^2 du. \end{aligned}$$

6. THE SAVINGS ACCOUNT AND ITS TRANSITION DENSITY

The savings account consists of the dollar wealth accumulated continuously at the short rate, given an initial deposit of one dollar at time zero. The value of the savings account at time t is given in (2.1). The following lemma leads to the formula for the savings account value under the Vasicek model.

Lemma 2. Let r_t satisfy the Vasicek SDE (3.3). Then

$$(6.1) \quad \int_t^{\bar{T}} r_s ds = r_t B(t, \bar{T}) + \bar{r}(\bar{T} - t - B(t, \bar{T})) + \sigma \int_t^{\bar{T}} B(u, \bar{T}) dZ_u,$$

where

$$(6.2) \quad B(t, \bar{T}) = \frac{1}{\kappa} (1 - \exp(-\kappa(\bar{T} - t))).$$

Proof. From (4.1) we have for $s \in [t, \bar{T}]$

$$(6.3) \quad r_s = r_t \exp(-\kappa(s - t)) + \bar{r}(1 - \exp(-\kappa(s - t))) + \sigma \int_t^s \exp(-\kappa(s - u)) dZ_u.$$

Integrating both sides with respect to s between t and \bar{T} gives

$$\begin{aligned}
(6.4) \quad & \int_t^{\bar{T}} r_s ds \\
&= \int_t^{\bar{T}} \left(r_t \exp(-\kappa(s-t)) + \bar{r}(1 - \exp(-\kappa(s-t))) \right. \\
&\quad \left. + \sigma \int_t^s \exp(-\kappa(s-u)) dZ_u \right) ds \\
&= \int_t^{\bar{T}} r_t \exp(-\kappa(s-t)) ds + \bar{r}(\bar{T}-t) + \bar{r} \frac{1}{\kappa} (\exp(-\kappa(\bar{T}-t)) - 1) \\
&\quad + \sigma \int_t^{\bar{T}} \int_t^s \exp(-\kappa(s-u)) dZ_u ds \\
&= r_t \frac{1}{\kappa} (1 - \exp(-\kappa(\bar{T}-t))) + \bar{r}(\bar{T}-t) - \frac{1}{\kappa} (1 - \exp(-\kappa(\bar{T}-t))) \\
&\quad + \sigma \frac{1}{\kappa} \int_t^{\bar{T}} (1 - \exp(-\kappa(\bar{T}-u))) dZ_u
\end{aligned}$$

which completes the proof. \square

A similar lemma applies to the Hull-White model.

Lemma 3. *Let r_t satisfy the Hull-White SDE (3.4). Then*

$$(6.5) \quad \int_t^{\bar{T}} r_s ds = r_t B(t, \bar{T}) + \int_t^{\bar{T}} B(u, \bar{T}) \{ \theta(u) + a(u)b \} du + \int_t^{\bar{T}} B(u, \bar{T}) \sigma(u) dZ_u,$$

where

$$(6.6) \quad B(t, \bar{T}) = \int_t^{\bar{T}} \exp \left\{ - \int_t^s a(\tau) d\tau \right\} ds.$$

The following proposition provides the formula for the savings account under the Vasicek short rate model.

Proposition 3. *Let r_t satisfy the Vasicek SDE (3.3). Then the SDE*

$$(6.7) \quad dB_t = r_t B_t dt$$

of the savings account B_t has the solution

$$(6.8) \quad B_{\bar{T}} = B_t \exp \left(r_t B(t, \bar{T}) + \bar{r}(\bar{T}-t - B(t, \bar{T})) + \sigma \int_t^{\bar{T}} B(u, \bar{T}) dZ_u \right)$$

where $B(t, \bar{T})$ is as in (6.2).

Proof. Combining (6.1) and (2.1) gives the formula for the savings account as

$$\begin{aligned}
(6.9) \quad & B_{\bar{T}} = B_t \exp \left(\int_t^{\bar{T}} r_s ds \right) \\
&= B_t \exp \left(r_t B(t, \bar{T}) + \bar{r}(\bar{T}-t - B(t, \bar{T})) + \sigma \int_t^{\bar{T}} B(u, \bar{T}) dZ_u \right),
\end{aligned}$$

which completes the proof. \square

From (6.8) we immediately see that the transition density function

$$(6.10) \quad p_B(t, B_t, \bar{T}, B_{\bar{T}})$$

of the savings account value is a lognormal density function.

Proposition 4. *Let r_t satisfy the Vasicek SDE (3.3). Then the transition density function of the savings account value $B_{\bar{T}}$ is*

$$(6.11) \quad p_B(t, B_t, \bar{T}, B_{\bar{T}}) = \frac{1}{B_{\bar{T}} \sqrt{2\pi v(t, \bar{T})}} \exp\left(-\frac{1}{2} \left(\log(B_{\bar{T}}/B_t) - m(t, \bar{T})\right)^2 / v(t, \bar{T})\right),$$

where

$$(6.12) \quad \begin{aligned} m(t, \bar{T}) &= r_t B(t, \bar{T}) + \bar{r}(\bar{T} - t - B(t, \bar{T})) \\ v(t, \bar{T}) &= \frac{\sigma^2}{\kappa^2} \left(\bar{T} - t - B(t, \bar{T}) - \frac{1}{2} \kappa B(t, \bar{T})^2 \right). \end{aligned}$$

Proof. From (6.8) we can write

$$(6.13) \quad B_{\bar{T}} = B_t \exp\left(m(t, \bar{T}) + \sqrt{v(t, \bar{T})} Z\right),$$

where

$$(6.14) \quad \begin{aligned} m(t, \bar{T}) &= r_t B(t, \bar{T}) + \bar{r}(\bar{T} - t - B(t, \bar{T})) \\ v(t, \bar{T}) &= \sigma^2 \int_t^{\bar{T}} B(u, \bar{T})^2 du \end{aligned}$$

and Z is a standard normal random variable. We can simplify the squared volatility $v(t, \bar{T})$ as follows

$$(6.15) \quad \begin{aligned} v(t, \bar{T}) &= \sigma^2 \int_t^{\bar{T}} \frac{1}{\kappa^2} (1 - \exp(-\kappa(\bar{T} - u)))^2 du \\ &= \frac{\sigma^2}{\kappa^2} \left((\bar{T} - t) - 2 \frac{1 - \exp(-\kappa(\bar{T} - t))}{\kappa} + \frac{1 - \exp(-2\kappa(\bar{T} - t))}{2\kappa} \right) \\ &= \frac{\sigma^2}{\kappa^2} \left((\bar{T} - t) - 2B(t, \bar{T}) + \frac{1 - (1 - \kappa B(t, \bar{T}))^2}{2\kappa} \right) \\ &= \frac{\sigma^2}{\kappa^2} \left(\bar{T} - t - B(t, \bar{T}) - \frac{1}{2} \kappa B(t, \bar{T})^2 \right) \end{aligned}$$

and we have the result. \square

Therefore, we can write the conditional distribution of the savings account value as

$$(6.16) \quad \log B_{\bar{T}} \sim N\left(\log B_t + m(t, \bar{T}), v(t, \bar{T})\right)$$

given B_t for $m(t, \bar{T})$ and $v(t, \bar{T})$ as in (6.12), where $N(m, v)$ denotes the Gaussian distribution with mean m and variance v .

Analogously, for the Hull-White model we can write the conditional distribution of the savings account value as

$$(6.17) \quad \log B_{\bar{T}} \sim N\left(\log B_t + m(t, \bar{T}), v(t, \bar{T})\right)$$

given B_t , where $m(t, \bar{T})$ and $v(t, \bar{T})$ are given by

$$(6.18) \quad \begin{aligned} m(t, \bar{T}) &= r_t \int_t^{\bar{T}} \exp \left\{ - \int_t^s a(\tau) d\tau \right\} ds \\ &\quad + \int_t^{\bar{T}} \left[\int_t^s \exp \left\{ - \int_u^s a(\tau) d\tau \right\} (\theta(u) + a(u)b) du \right] ds \\ v(t, \bar{T}) &= \int_t^{\bar{T}} \left[\int_u^{\bar{T}} \exp \left\{ - \int_u^s a(\tau) d\tau \right\} ds \right]^2 \sigma(u)^2 du. \end{aligned}$$

The transition density of the savings account allows us to calculate the zero coupon bond price in the following section.

7. ZCB PRICE

In the following lemma we calculate the \bar{T} -maturity zero coupon bond price $G_{\bar{T}}(t)$, given in (2.6) and which can be rewritten as

$$(7.1) \quad G_{\bar{T}}(t) = E \left(\frac{B_t}{B_{\bar{T}}} \middle| \mathcal{A}_t \right).$$

Lemma 4. *Let r_t satisfy the Vasicek SDE (3.3). Then for time $t \in [0, \bar{T}]$ the \bar{T} -maturity ZCB price is*

$$(7.2) \quad G_{\bar{T}}(t) = A(t, \bar{T}) \exp(-r_t B(t, \bar{T})),$$

where

$$(7.3) \quad B(t, \bar{T}) = \frac{1 - \exp(-\kappa(\bar{T} - t))}{\kappa}$$

and

$$(7.4) \quad A(t, \bar{T}) = \exp \left(\left(\bar{r} - \frac{\sigma^2}{2\kappa^2} \right) (B(t, \bar{T}) - \bar{T} + t) - \frac{\sigma^2}{4\kappa} B(t, \bar{T})^2 \right).$$

Proof. From (6.16)

$$(7.5) \quad \log B_{\bar{T}} \sim N(\log B_t + m(t, \bar{T}), v(t, \bar{T}))$$

given B_t and using (7.1) we have

$$\begin{aligned}
(7.6) \quad G_{\bar{T}}(t) &= E \left(\frac{B_t}{B_{\bar{T}}} \middle| \mathcal{A}_t \right) \\
&= B_t E \left(\exp(-\log B_{\bar{T}}) \middle| \mathcal{A}_t \right) \\
&= B_t \exp(-E(\log B_{\bar{T}} | \mathcal{A}_t) + \frac{1}{2} \text{VAR}(B_{\bar{T}} | \mathcal{A}_t)) \\
&= B_t \exp(-\log B_t - m(t, \bar{T}) + \frac{1}{2} v(t, \bar{T})) \\
&= \exp(-m(t, \bar{T}) + \frac{1}{2} v(t, \bar{T})) \\
&= \exp \left(-r_t B(t, \bar{T}) - \bar{r}(\bar{T} - t - B(t, \bar{T})) \right. \\
&\quad \left. + \frac{\sigma^2}{2\kappa^2} \left(\bar{T} - t - B(t, \bar{T}) - \frac{1}{2} \kappa B(t, \bar{T})^2 \right) \right) \\
&= \exp(-r_t B(t, \bar{T})) \exp \left((-\bar{r} + \frac{\sigma^2}{2\kappa^2})(\bar{T} - t - B(t, \bar{T})) - \frac{\sigma^2}{4\kappa} B(t, \bar{T})^2 \right)
\end{aligned}$$

which is the result. \square

A similar result can be proven for the Hull-White short rate model.

Lemma 5. *Let r_t satisfy the Hull-White SDE (3.4). Then for time $t \in [0, \bar{T}]$ the \bar{T} -maturity ZCB price is*

$$(7.7) \quad G_{\bar{T}}(t) = A(t, \bar{T}) \exp(-r_t B(t, \bar{T})),$$

where

$$(7.8) \quad B(t, \bar{T}) = \int_t^{\bar{T}} \exp \left\{ - \int_t^s a(\tau) d\tau \right\} ds$$

and

$$\begin{aligned}
(7.9) \quad A(t, \bar{T}) &= \exp \left(- \int_t^{\bar{T}} \left[\int_t^s \exp \left\{ - \int_u^s a(\tau) d\tau \right\} (\theta(u) + a(u)b) du \right] ds \right. \\
&\quad \left. + \frac{1}{2} \int_t^{\bar{T}} B(u, \bar{T})^2 \sigma(u)^2 du \right).
\end{aligned}$$

8. BOND YIELDS AND FORWARD RATES

We investigate the asymptotic level of the yield curve under the Vasicek model. As a corollary of Lemma 4 we calculate the \bar{T} -maturity ZCB yield $h_{\bar{T}}(t)$, as given in

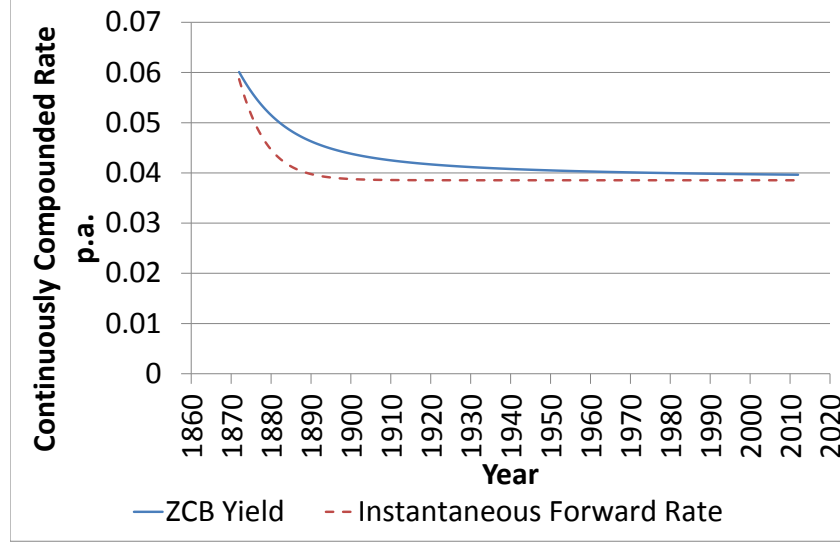
$$(8.1) \quad h_{\bar{T}}(t) = -\frac{1}{\bar{T} - t} \log G_{\bar{T}}(t),$$

as $\bar{T} \rightarrow \infty$, which we call the long ZCB yield.

Corollary 3. *Let r_t satisfy the Vasicek SDE (3.3). Then the long ZCB yield is*

$$(8.2) \quad h_{\infty}(t) = \bar{r} - \frac{\sigma^2}{2\kappa^2}.$$

FIGURE 5. Zero coupon yield curve under the Vasicek model based at 1871.



Proof. From (8.1), the ZCB yield is given by the formula

$$\begin{aligned}
 (8.3) \quad h_\infty(t) &= - \lim_{\bar{T} \rightarrow \infty} \frac{1}{\bar{T} - t} \log G_{\bar{T}}(t) \\
 &= \lim_{\bar{T} \rightarrow \infty} \frac{1}{\bar{T} - t} (-\log A(t, \bar{T}) + r_t B(t, \bar{T})) \\
 &= \lim_{\bar{T} \rightarrow \infty} r_t \frac{B(t, \bar{T})}{\bar{T} - t} - \left(\bar{r} - \frac{\sigma^2}{2\kappa^2} \right) \frac{B(t, \bar{T}) - \bar{T} + t}{\bar{T} - t} + \frac{\sigma^2}{4\kappa(\bar{T} - t)} B(t, \bar{T})^2.
 \end{aligned}$$

But $\lim_{\bar{T} \rightarrow \infty} B(t, \bar{T}) = \frac{1}{\kappa}$ and, therefore, the long ZCB yield simplifies to $\bar{r} - \frac{\sigma^2}{2\kappa^2}$. \square

In Figure 5 the continuously compounded yield curve is plotted as at the time of 1871. We have an inverted yield curve and this portends an economic recession because decreasing forward rates indicate expectations of low inflation and low economic growth, as discussed in Harvey [1991].

For the Hull-White model, we have the following theorem which gives the long ZCB yield under suitable conditions on the functions $\theta(u)$, $a(u)$ and $\sigma(u)$ in (3.4).

Theorem 1. *Let r_t satisfy the Hull-White SDE (3.4). Suppose further that as $u \rightarrow \infty$ we have*

$$\begin{aligned}
 (8.4) \quad a(u) &\rightarrow \bar{a} > 0 \\
 \theta(u) &\rightarrow \bar{\theta} \in \mathbb{R} \\
 \sigma(u) &\rightarrow \bar{\sigma} > 0.
 \end{aligned}$$

Then the long ZCB yield is

$$(8.5) \quad h_\infty(t) = b + \frac{\bar{\theta}}{\bar{a}} - \frac{\bar{\sigma}^2}{2\bar{a}^2}.$$

Proof. We first show that the integral

$$(8.6) \quad I = \int_t^{\bar{T}} \int_t^s \exp \left\{ - \int_u^s a(\tau) d\tau \right\} \theta(u) du ds$$

is asymptotic to $\frac{\bar{\theta}}{\bar{a}}(\bar{T} - t)$ as $\bar{T} \rightarrow \infty$. We let ϵ be a fixed positive number less than one and choose the number $T_\epsilon > t$ such that for $u > T_\epsilon$, $|a(u) - \bar{a}| < \epsilon$ and $|\theta(u) - \bar{\theta}| < \epsilon$. We also choose the number \bar{T}_ϵ such that

$$(8.7) \quad \max \left\{ \frac{1}{\bar{T}_\epsilon - t}, \frac{T_\epsilon - t}{\bar{T}_\epsilon - t} \right\} = \epsilon.$$

We note that since $\epsilon < 1$ we must have $\bar{T}_\epsilon > T_\epsilon$. Also, let the function $a(u)$ have lower and upper bounds L_a and U_a for $u \in [0, \infty)$. It is evident from the positivity of $a(u)$ that $0 < L_a \leq U_a$. Finally, let $\theta(u)$ have lower and upper bounds L_θ and U_θ for $u \in [0, \infty)$.

Now, for $\bar{T} > \bar{T}_\epsilon$ we show that the integral I in (8.6) is close to $\frac{\bar{\theta}}{\bar{a}}(\bar{T} - t)$. Without loss of generality, we assume that $\theta(u) \geq 0$. Indeed, if $L_\theta < 0$ we can express the integral I as a sum of two integrals, that is,

$$(8.8) \quad I = \int_t^{\bar{T}} \int_t^s \exp \left\{ - \int_u^s a(\tau) d\tau \right\} (\theta(u) - L_\theta) du ds + L_\theta \int_t^{\bar{T}} \int_t^s \exp \left\{ - \int_u^s a(\tau) d\tau \right\} du ds,$$

and apply the following reasoning to each of the integrals, obtaining the desired result.

Partitioning the region of integration into three subregions, we write

$$(8.9) \quad I = \left\{ \int_t^{T_\epsilon} \int_t^s + \int_{T_\epsilon}^{\bar{T}} \int_{T_\epsilon}^s + \int_{T_\epsilon}^{\bar{T}} \int_t^{T_\epsilon} \right\} \exp \left\{ - \int_u^s a(\tau) d\tau \right\} \theta(u) du ds = I_A + I_B + I_C.$$

Using the lower and upper bounds of $a(u)$ and $\theta(u)$ on $[0, \infty)$ we have

$$(8.10) \quad \int_t^{T_\epsilon} \int_t^s \exp \left\{ - (s - u) U_a \right\} L_\theta du ds < I_A < \int_t^{T_\epsilon} \int_t^s \exp \left\{ - (s - u) L_a \right\} U_\theta du ds,$$

which gives the inequalities

$$(8.11) \quad \begin{aligned} \frac{L_\theta}{U_a}(T_\epsilon - t) - \frac{L_\theta}{U_a^2} &< \frac{L_\theta}{U_a}(T_\epsilon - t) - \frac{L_\theta}{U_a^2}(1 - \exp\{-U_a(T_\epsilon - t)\}) < I_A \\ I_A &< \frac{U_\theta}{L_a}(T_\epsilon - t) - \frac{U_\theta}{L_a^2}(1 - \exp\{-L_a(T_\epsilon - t)\}) < \frac{U_\theta}{L_a}(T_\epsilon - t). \end{aligned}$$

Similarly, using the lower and upper bounds of $a(u)$ and $\theta(u)$ on $[T_\epsilon, \infty)$, that is,

$$(8.12) \quad \begin{aligned} \bar{a} - \epsilon &< a(u) < \bar{a} + \epsilon \\ \bar{\theta} - \epsilon &< \theta(u) < \bar{\theta} + \epsilon, \end{aligned}$$

we have the inequalities

$$(8.13) \quad \frac{\bar{\theta} - \epsilon}{\bar{a} + \epsilon}(\bar{T} - T_\epsilon) - \frac{\bar{\theta} - \epsilon}{(\bar{a} + \epsilon)^2} < I_B < \frac{\bar{\theta} + \epsilon}{\bar{a} - \epsilon}(\bar{T} - T_\epsilon).$$

Finally, we have the inequalities for I_C

$$(8.14) \quad 0 \leq I_C < \int_{T_\epsilon}^{\bar{T}} \int_t^{T_\epsilon} \exp\{-L_a(s-u)\} U_\theta du ds < \frac{U_\theta}{L_a^2}.$$

Combining Inequalities (8.11), (8.13) and (8.14) gives, for $\bar{T} > \bar{T}_\epsilon$,

$$(8.15) \quad \begin{aligned} & \bar{a} \frac{L_\theta T_\epsilon - t}{U_a \bar{T} - t} - \frac{\bar{a}}{\bar{T} - t} \frac{L_\theta}{U_a^2} + \frac{\bar{\theta} - \epsilon}{1 - \epsilon/\bar{a}} \frac{\bar{T} - T_\epsilon}{\bar{T} - t} - \frac{\bar{a}}{\bar{T} - t} \frac{\bar{\theta} - \epsilon}{(\bar{a} + \epsilon)^2} \\ & < \frac{\bar{a}}{\bar{T} - t} (I_A + I_B + I_C) = \frac{\bar{a}}{\bar{T} - t} I \\ & < \bar{a} \frac{U_\theta T_\epsilon - t}{L_a \bar{T} - t} + \frac{\bar{\theta} + \epsilon}{1 - \epsilon/\bar{a}} \frac{\bar{T} - T_\epsilon}{\bar{T} - t} + \frac{\bar{a}}{\bar{T} - t} \frac{U_\theta}{L_a^2}. \end{aligned}$$

Making use of our choice of \bar{T}_ϵ in (8.7) the above inequalities simplify to

$$(8.16) \quad -\bar{a} \frac{L_\theta}{U_a^2} \epsilon + \frac{\bar{\theta} - \epsilon}{1 - \epsilon/\bar{a}} (1 - \epsilon) - \bar{a} \epsilon \frac{\bar{\theta} - \epsilon}{(\bar{a} + \epsilon)^2} < \frac{\bar{a}}{\bar{T} - t} I < \bar{a} \frac{U_\theta}{L_a} \epsilon + \frac{\bar{\theta} + \epsilon}{1 - \epsilon/\bar{a}} + \bar{a} \epsilon \frac{U_\theta}{L_a^2},$$

for $\bar{T} > \bar{T}_\epsilon$. Thus, (8.16) demonstrates that $\frac{\bar{a}}{\bar{T} - t} I$ can be made arbitrarily close to $\bar{\theta}$, as $\bar{T} \rightarrow \infty$. The proofs of the limits

$$(8.17) \quad \begin{aligned} & \lim_{\bar{T} \rightarrow \infty} \frac{1}{\bar{T} - t} \int_t^{\bar{T}} \int_t^s \exp\left\{-\int_u^s a(\tau) d\tau\right\} a(u) b du ds = b \\ & \lim_{\bar{T} \rightarrow \infty} \frac{1}{\bar{T} - t} \int_t^{\bar{T}} \left[\int_u^{\bar{T}} \exp\left\{-\int_u^s a(\tau) d\tau\right\} ds \right]^2 \sigma(u)^2 du = \frac{\bar{\sigma}^2}{\bar{a}} \end{aligned}$$

are essentially the same. Therefore, as $\bar{T} \rightarrow \infty$, the long ZCB yield simplifies to $b + \frac{\bar{\theta}}{\bar{a}} - \frac{\bar{\sigma}^2}{2\bar{a}^2}$. \square

We calculate the forward rate $g_{\bar{T}}(t)$, given by

$$(8.18) \quad g_{\bar{T}}(t) = -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t).$$

Lemma 6. For time $t \in [0, \bar{T}]$ the forward rate is computed to be

$$(8.19) \quad g_{\bar{T}}(t) = (r_t - \bar{r}) \exp(-\kappa(\bar{T} - t)) + \bar{r} - \frac{\sigma^2}{2\kappa^2} \left(1 - \exp(-\kappa(\bar{T} - t))\right)^2.$$

Proof. Using (8.18) and (7.2) we have

$$(8.20) \quad \begin{aligned} g_{\bar{T}}(t) &= -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t) \\ &= -\frac{\partial}{\partial \bar{T}} \left\{ - (r_t - \bar{r}) \frac{(1 - \exp(-\kappa(\bar{T} - t)))}{\kappa} - \bar{r}(\bar{T} - t) \right. \\ & \quad \left. + \frac{\sigma^2}{2\kappa^2} \left((\bar{T} - t) - 2 \frac{1 - \exp(-\kappa(\bar{T} - t))}{\kappa} + \frac{1 - \exp(-2\kappa(\bar{T} - t))}{2\kappa} \right) \right\} \\ &= (r_t - \bar{r}) \exp(-\kappa(\bar{T} - t)) + \bar{r} \\ & \quad - \frac{\sigma^2}{2\kappa^2} \left(1 - 2 \exp(-\kappa(\bar{T} - t)) + \exp(-2\kappa(\bar{T} - t)) \right) \end{aligned}$$

and simplifying gives the result. \square

As a corollary of this lemma we calculate directly the asymptotic instantaneous forward rate.

Corollary 4. *For the Vasicek short rate model, the asymptotic instantaneous forward rate is*

$$(8.21) \quad g_\infty(t) = \bar{r} - \frac{\sigma^2}{2\kappa^2}.$$

In Figure 5 the instantaneous forward rate $g_{\bar{T}}$ is plotted and can be seen to be asymptotic to $g_\infty(t) = 0.0385$ based upon the parameters in (5.4).

9. EXPECTATIONS INVOLVING $G_{\bar{T}}(t)$

Motivated by our goal of pricing call and put options on zero coupon bonds, we seek formulae for the following expectations

$$(9.1) \quad \begin{aligned} f_1(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) G_{\bar{T}}(T) \mathbf{1}_{G_{\bar{T}}(T) > K} \middle| \mathcal{A}_t\right) \\ f_2(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) G_{\bar{T}}(T) \mathbf{1}_{G_{\bar{T}}(T) \leq K} \middle| \mathcal{A}_t\right) \\ f_3(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) \mathbf{1}_{G_{\bar{T}}(T) > K} \middle| \mathcal{A}_t\right) \\ f_4(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) \mathbf{1}_{G_{\bar{T}}(T) \leq K} \middle| \mathcal{A}_t\right) \\ f_5(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) (G_{\bar{T}}(T) - K)^+ \middle| \mathcal{A}_t\right) \\ f_6(t, T, K, \bar{T}) &= E\left(\exp\left(-\int_t^T r_s ds\right) (K - G_{\bar{T}}(T))^+ \middle| \mathcal{A}_t\right), \end{aligned}$$

where $0 \leq t < T < \bar{T}$ and $K > 0$. These expectations correspond to prices of various call and put options on zero coupon bonds under the Vasicek short rate model.

It is well known that the Vasicek short rate model is an example of a Gaussian interest rate model and that for such models the prices of call options on zero coupon bonds employ the Black-Scholes option pricing formula. We establish that the Black-Scholes formula applies when performing actuarial pricing of contingent claims, employing the following four lemmas which culminate in three subsequent theorems.

Lemma 7. *We have*

$$(9.2) \quad \begin{aligned} E(\exp(\alpha Z)) &= \exp\left(\frac{1}{2}\alpha^2\right) \\ E(\exp(\alpha Z) \mathbf{1}_{Z > z}) &= \exp\left(\frac{1}{2}\alpha^2\right)(1 - N(z - \alpha)) \\ E(\exp(\alpha Z) \mathbf{1}_{Z \leq z}) &= \exp\left(\frac{1}{2}\alpha^2\right)N(z - \alpha) \end{aligned}$$

Proof. We have

$$\begin{aligned}
(9.3) \quad E(\exp(\alpha Z)) &= \int_{-\infty}^{\infty} \exp(\alpha u) n(u) du \\
&= \int_{-\infty}^{\infty} \exp\left(\frac{1}{2}\alpha^2\right) n(u - \alpha) du \\
&= \exp\left(\frac{1}{2}\alpha^2\right) \int_{-\infty}^{\infty} n(v) dv \\
&= \exp\left(\frac{1}{2}\alpha^2\right).
\end{aligned}$$

Next we have

$$\begin{aligned}
(9.4) \quad E(\exp(\alpha Z) \mathbf{1}_{Z>z}) &= \int_z^{\infty} \exp(\alpha u) n(u) du \\
&= \int_z^{\infty} \exp\left(\frac{1}{2}\alpha^2\right) n(u - \alpha) du \\
&= \exp\left(\frac{1}{2}\alpha^2\right) \int_{z-\alpha}^{\infty} n(v) dv \\
&= \exp\left(\frac{1}{2}\alpha^2\right) E(\mathbf{1}_{Z>z-\alpha}) \\
&= \exp\left(\frac{1}{2}\alpha^2\right) (1 - N(z - \alpha)),
\end{aligned}$$

which is the second result. The third result is obtained by transposing the identity

$$(9.5) \quad E(\exp(\alpha Z) \mathbf{1}_{Z>z}) + E(\exp(\alpha Z) \mathbf{1}_{Z\leq z}) = E(\exp(\alpha Z))$$

and applying the first two results. \square

Lemma 8. *Let Y be a normally distributed random variable. Then for any real number y we have*

$$(9.6) \quad E(\exp(Y) \mathbf{1}_{Y\leq y}) = E(\exp(Y)) \times E(\mathbf{1}_{Y\leq y - VAR(Y)})$$

and

$$(9.7) \quad E(\exp(Y) \mathbf{1}_{Y>y}) = E(\exp(Y)) \times E(\mathbf{1}_{Y>y - VAR(Y)}).$$

Proof. Let us write the random variable Y in the form $Y = \mu + \sigma Z$, where Z is a standard normal random variable and σ is a positive real number. Clearly, $E(Y) = \mu$ and $VAR(Y) = \sigma^2$. Also $E(\exp(Y)) = \exp(\mu + \frac{1}{2}\sigma^2)$. Then

$$\begin{aligned}
(9.8) \quad E(\exp(Y) \mathbf{1}_{Y\leq y}) &= E(\exp(\mu + \sigma Z) \mathbf{1}_{\mu + \sigma Z \leq y}) \\
&= \exp(\mu) E(\exp(\sigma Z) \mathbf{1}_{Z \leq (y-\mu)/\sigma})
\end{aligned}$$

and from Lemma 7 we have

$$\begin{aligned}
(9.9) \quad E(\exp(Y) \mathbf{1}_{Y\leq y}) &= \exp(\mu) \exp\left(\frac{1}{2}\sigma^2\right) E(\mathbf{1}_{Z \leq (y-\mu)/\sigma - \sigma}) \\
&= E(\exp(Y)) \times E(\mathbf{1}_{Z \leq (y-\mu)/\sigma - \sigma}) \\
&= E(\exp(Y)) \times E(\mathbf{1}_{Y \leq y - \sigma^2})
\end{aligned}$$

as required. Also the second equality emerges after applying the relation

$$(9.10) \quad E(\exp(Y) \mathbf{1}_{Y>y}) = E(\exp(Y)(1 - \mathbf{1}_{Y\leq y})) = E(\exp(Y)) - E(\exp(Y) \mathbf{1}_{Y\leq y})$$

to the first equality. \square

In the following lemma we state an extension of the above lemma.

Lemma 9. *Let Y_1 and Y_2 be normally distributed random variables. Then for any real number y ,*

$$(9.11) \quad E(\exp(Y_1)\mathbf{1}_{Y_2 \leq y}) = E(\exp(Y_1)) \times E(\mathbf{1}_{Y_2 \leq y - COV(Y_1, Y_2)}).$$

Also we have

$$(9.12) \quad E(\exp(Y_1)\mathbf{1}_{Y_2 > y}) = E(\exp(Y_1)) \times E(\mathbf{1}_{Y_2 > y - COV(Y_1, Y_2)}).$$

Proof. We let

$$(9.13) \quad Y'_2 = Y_2 - \beta Y_1,$$

where $\beta = COV(Y_1, Y_2)/VAR(Y_1)$. This allows us to write Y_2 as a linear combination of two uncorrelated random variables Y_1 and Y'_2 as follows:

$$(9.14) \quad Y_2 = \beta Y_1 + Y'_2.$$

If $\beta = 0$, then Y_1 and Y_2 are uncorrelated and, because both are normally distributed random variables, are therefore independent which gives, by Lemma 8, the result. Henceforth we assume $\beta \neq 0$ and we have, by Lemma 8,

$$(9.15) \quad \begin{aligned} E(\exp(Y_1)\mathbf{1}_{Y_2 \leq y}) &= E(\exp(Y_1)\mathbf{1}_{\beta Y_1 + Y'_2 \leq y}) \\ &= E(E(\exp(Y_1)\mathbf{1}_{\beta Y_1 + Y'_2 \leq y} | Y'_2)) \\ &= E(E(\exp(Y_1)\mathbf{1}_{Y_1 \leq \frac{1}{\beta}(y - Y'_2)} | Y'_2)) \text{ for } \beta > 0. \end{aligned}$$

We remark that for $\beta < 0$ the inequality is reversed in the indicator function above, yet an identical result to that which follows is obtained. We apply Lemma 8 to evaluate the inner expectation, giving

$$(9.16) \quad \begin{aligned} &E(E(\exp(Y_1)\mathbf{1}_{Y_1 \leq \frac{1}{\beta}(y - Y'_2)} | Y'_2)) \\ &= E(E(\exp(Y_1))E(\mathbf{1}_{Y_1 \leq \frac{1}{\beta}(y - Y'_2) - VAR(Y_1)} | Y'_2)) \\ &= E(\exp(Y_1))E(E(\mathbf{1}_{\beta Y_1 + Y'_2 \leq y - \beta VAR(Y_1)} | Y'_2)) \\ &= E(\exp(Y_1))E(\mathbf{1}_{Y_2 \leq y - \beta VAR(Y_1)}) \\ &= E(\exp(Y_1))E(\mathbf{1}_{Y_2 \leq y - COV(Y_1, Y_2)}) \end{aligned}$$

which is the first equality. The second equality also follows similarly. \square

We can readily prove the formulae for ZCB call and put options using Lemma 9 when the integral of the short rate is a normally distributed random variable whose variance parameter is a deterministic function, that is when the following condition holds:

Condition 1. *The integral $\int_t^T r_s ds$ is normally distributed, that is,*

$$(9.17) \quad \int_t^T r_s ds \sim N(m, v),$$

where the parameter v is a deterministic function involving the parameters \bar{r} , κ , σ , t and T .

This condition is satisfied by the Ho-Lee short rate model, the Hull-White short rate model and various extended versions of these. Therefore, our lemmas apply to these models, which result in a proof of the Black-Scholes formula for options on zero coupon bonds under each of these models.

Lemma 10. *Let r_t be a process for the short rate which satisfies Condition 1 and let*

$$(9.18) \quad G_{\bar{T}}(T) = E \left(\exp \left(- \int_T^{\bar{T}} r_s ds \right) \middle| \mathcal{A}_T \right).$$

Then the random variable L conditional on information up to time t , given by

$$(9.19) \quad L = \log G_{\bar{T}}(T),$$

is normally distributed whose expected value satisfies

$$(9.20) \quad E(L|\mathcal{A}_t) = \log G_{\bar{T}}(t)/G_T(t) - \frac{1}{2} \text{VAR}(L|\mathcal{A}_t) + \text{COV} \left(L, \int_t^T r_s ds \middle| \mathcal{A}_t \right)$$

and whose variance $\text{VAR}(L|\mathcal{A}_t)$ satisfies

$$(9.21) \quad \text{VAR}(L|\mathcal{A}_t) = \text{VAR} \left(E \left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right) \middle| \mathcal{A}_t \right).$$

Proof. Because $\int_0^t r_s ds$ is normally distributed we have

$$(9.22) \quad \begin{aligned} G_{\bar{T}}(T) &= E \left(\exp \left(- \int_T^{\bar{T}} r_s ds \right) \middle| \mathcal{A}_T \right) \\ &= E \left(\exp \left(- \int_t^{\bar{T}} r_s ds + \int_t^T r_s ds \right) \middle| \mathcal{A}_T \right) \\ &= \exp \left(E \left(- \int_t^{\bar{T}} r_s ds + \int_t^T r_s ds \middle| \mathcal{A}_T \right) \right) \\ &\quad + \frac{1}{2} \text{VAR} \left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right), \end{aligned}$$

where we have used Condition 1, namely that $\text{VAR} \left(\int_t^T r_s ds \middle| \mathcal{A}_T \right) = 0$, and the properties of the lognormal distribution. Therefore, the conditional random variable L given the information available at time t is given by

$$(9.23) \quad L = \log G_{\bar{T}}(T) = -E \left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right) + \int_t^T r_s ds + \frac{1}{2} \text{VAR} \left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right)$$

and is normally distributed.

Its expected value is

$$(9.24) \quad \begin{aligned} E(L|\mathcal{A}_t) &= -E \left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t \right) + E \left(\int_t^T r_s ds \middle| \mathcal{A}_t \right) \\ &\quad + \frac{1}{2} E \left(\text{VAR} \left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right) \middle| \mathcal{A}_t \right), \end{aligned}$$

which simplifies to

$$(9.25) \quad E(L|\mathcal{A}_t) = -E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t\right) + E\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right) \\ + \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right),$$

because the variance $VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right)$ is deterministic, as demonstrated by

$$(9.26) \quad VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) = VAR\left(\int_t^T r_s ds + \int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \\ = VAR\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \\ = v(T, \bar{T}).$$

To simplify (9.25) we note that

$$(9.27) \quad \log G_{\bar{T}}(t) = -E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t\right) + \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t\right) \\ \log G_T(t) = -E\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right) + \frac{1}{2}VAR\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right)$$

and that, by virtue of the Law of Total Variance,

$$(9.28) \quad VAR(X) = VAR(E(X|Y)) + E(VAR(X|Y)),$$

we have

$$(9.29) \quad VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t\right) = VAR\left(E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right) + VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right).$$

Therefore, we can rewrite (9.25) as

$$(9.30) \quad E(L|\mathcal{A}_t) = \log G_{\bar{T}}(t) - \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_t\right) \\ - \log G_T(t) + \frac{1}{2}VAR\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right) \\ + \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \\ = \log G_{\bar{T}}(t)/G_T(t) \\ - \frac{1}{2}VAR\left(E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right) - \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \\ + \frac{1}{2}VAR\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right) + \frac{1}{2}VAR\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \\ = \log G_{\bar{T}}(t)/G_T(t) \\ - \frac{1}{2}VAR\left(E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right) + \frac{1}{2}VAR\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right).$$

Transposing (9.23) gives

$$(9.31) \quad E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) = -L + \int_t^T r_s ds + \frac{1}{2} \text{VAR}\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right)$$

and taking the variance of both sides gives

$$(9.32) \quad \text{VAR}\left(E\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right) = \text{VAR}(L | \mathcal{A}_t) + \text{VAR}\left(\int_t^T r_s ds \middle| \mathcal{A}_t\right) - 2\text{COV}\left(L, \int_t^T r_s ds \middle| \mathcal{A}_t\right).$$

Substituting this variance formula into (9.30) gives (9.20). The formula for the variance (9.21) is easily deduced by rewriting (9.23) as

$$(9.33) \quad L = -E\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) + \frac{1}{2} \text{VAR}\left(\int_t^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right)$$

and taking variances of both sides. \square

Theorem 2. *Let r_t be a process for the short rate which satisfies Condition 1. Then the formulae for the expectations f_1 and f_2 in (9.1) are given by*

$$(9.34) \quad \begin{aligned} f_1(t, T, K, \bar{T}) &= G_{\bar{T}}(t)N(d_1) \\ f_2(t, T, K, \bar{T}) &= G_{\bar{T}}(t)(1 - N(d_1)), \end{aligned}$$

where

$$(9.35) \quad \begin{aligned} d_1 &= \frac{1}{2}\sigma_G + \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= \text{VAR}\left(E\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right). \end{aligned}$$

Proof. The price of the asset-or-nothing call option on $G_{\bar{T}}$ is

$$(9.36) \quad \begin{aligned} f_1(t, T, K, \bar{T}) &= E\left(\exp\left\{-\int_t^T r_s ds\right\} G_{\bar{T}}(T) \mathbf{1}_{G_{\bar{T}}(T) > K} \middle| \mathcal{A}_t\right) \\ &= E\left(\exp\left\{-\int_t^T r_s ds\right\} E\left(\exp\left\{-\int_T^{\bar{T}} r_s ds\right\} \middle| \mathcal{A}_T\right) \mathbf{1}_{L > \log K} \middle| \mathcal{A}_t\right) \\ &= E\left(\exp\left\{-\int_t^{\bar{T}} r_s ds\right\} \mathbf{1}_{L > \log K} \middle| \mathcal{A}_t\right). \end{aligned}$$

We can apply Lemma 9 to the right hand side of (9.36) to give

$$(9.37) \quad \begin{aligned} f_1(t, T, K, \bar{T}) &= E\left(\exp\left\{-\int_t^{\bar{T}} r_s ds\right\} \middle| \mathcal{A}_t\right) \times E\left(\mathbf{1}_{L > \log K - \text{COV}(-\int_t^{\bar{T}} r_s ds, L | \mathcal{A}_t)} \middle| \mathcal{A}_t\right) \\ &= G_{\bar{T}}(t) E(\mathbf{1}_{Z > z_1}) \end{aligned}$$

for a standard normal random variable Z where

$$(9.38) \quad z_1 = \frac{1}{\sqrt{\text{VAR}(L)}} \left(\log K + \text{COV}\left(\int_t^{\bar{T}} r_s ds, L \middle| \mathcal{A}_t\right) - E(L) \right).$$

The expression $\log K + COV\left(\int_t^{\bar{T}} r_s ds, L \mid \mathcal{A}_t\right) - E(L)$ can be simplified using (9.30) to give

$$\begin{aligned}
(9.39) \quad & \log KG_T(t)/G_{\bar{T}}(t) + COV\left(\int_t^{\bar{T}} r_s ds, L \mid \mathcal{A}_t\right) \\
& + \frac{1}{2}VAR(L \mid \mathcal{A}_t) - COV\left(L, \int_t^T r_s ds \mid \mathcal{A}_t\right) \\
& = \log KG_T(t)/G_{\bar{T}}(t) + COV\left(\int_T^{\bar{T}} r_s ds, L \mid \mathcal{A}_t\right) \\
& + \frac{1}{2}VAR(L \mid \mathcal{A}_t).
\end{aligned}$$

From (9.33) we have

$$\begin{aligned}
(9.40) \quad & COV\left(\int_T^{\bar{T}} r_s ds, L \mid \mathcal{A}_t\right) \\
& = -COV\left(\int_T^{\bar{T}} r_s ds, E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right) \mid \mathcal{A}_t\right) \\
& = -E\left(\int_T^{\bar{T}} r_s ds \times E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right) \mid \mathcal{A}_t\right) \\
& + E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_t\right) \times E\left(E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right) \mid \mathcal{A}_t\right).
\end{aligned}$$

Using the law of total covariance, we have

$$\begin{aligned}
(9.41) \quad & COV\left(\int_T^{\bar{T}} r_s ds, L \mid \mathcal{A}_t\right) \\
& = -E\left(E\left(\int_T^{\bar{T}} r_s ds \times E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right) \mid \mathcal{A}_T\right) \mid \mathcal{A}_t\right) + \left\{E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_t\right)\right\}^2 \\
& = -E\left(\left\{E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right)\right\}^2 \mid \mathcal{A}_t\right) + \left\{E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_t\right)\right\}^2 \\
& = -VAR\left(E\left(\int_T^{\bar{T}} r_s ds \mid \mathcal{A}_T\right) \mid \mathcal{A}_t\right) \\
& = -VAR(L \mid \mathcal{A}_t).
\end{aligned}$$

Thus (9.39) becomes

$$(9.42) \quad \log KG_T(t)/G_{\bar{T}}(t) - \frac{1}{2}VAR(L \mid \mathcal{A}_t)$$

and, therefore, (9.38) becomes

$$(9.43) \quad z_1 = \frac{1}{\sqrt{VAR(L)}} \left(\log KG_T(t)/G_{\bar{T}}(t) - \frac{1}{2}VAR(L) \right).$$

Thus

$$(9.44) \quad E(\mathbf{1}_{Z > z_1}) = 1 - N(z_1) = N(-z_1) = N(d_1),$$

where

$$(9.45) \quad d_1 = \frac{1}{\sqrt{VAR(L)}} \left(\log G_{\bar{T}}(t)/KG_T(t) + \frac{1}{2}VAR(L) \right),$$

as specified in statement of the lemma. The formula for the asset-or-nothing binary put option is derived using call-put parity. \square

Theorem 3. *Let r_t be a process for the short rate which satisfies Condition 1. Then the formulae for the expectations f_3 and f_4 in (9.1) are given by*

$$(9.46) \quad \begin{aligned} f_3(t, T, K, \bar{T}) &= G_T(t)N(d_2) \\ f_4(t, T, K, \bar{T}) &= G_T(t)(1 - N(d_2)), \end{aligned}$$

where

$$(9.47) \quad \begin{aligned} d_2 &= -\frac{1}{2}\sigma_G + \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= VAR \left(E \left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right) \middle| \mathcal{A}_t \right). \end{aligned}$$

Proof. The price of the cash-or-nothing call option on $G_{\bar{T}}$ is

$$(9.48) \quad f_3(t, T, K, \bar{T}) = E \left(\exp \left\{ - \int_t^T r_s ds \right\} \mathbf{1}_{G_{\bar{T}}(T) > K} \middle| \mathcal{A}_t \right).$$

We can apply Lemma 9 to the right hand side of (9.48) to give

$$(9.49) \quad \begin{aligned} f_3(t, T, K, \bar{T}) &= E \left(\exp \left\{ - \int_t^T r_s ds \right\} \middle| \mathcal{A}_t \right) \times E \left(\mathbf{1}_{L > \log K - COV(-\int_t^T r_s ds, L | \mathcal{A}_t)} \middle| \mathcal{A}_t \right) \\ &= G_T(t)E(\mathbf{1}_{Z > z_2}) \end{aligned}$$

for a standard normal random variable Z where

$$(9.50) \quad z_2 = \frac{1}{\sqrt{VAR(L)}} \left(\log K + COV \left(\int_t^T r_s ds, L \middle| \mathcal{A}_t \right) - E(L) \right).$$

The expression $\log K + COV \left(\int_t^T r_s ds, L \middle| \mathcal{A}_t \right) - E(L)$ can be simplified using (9.30) to give

$$(9.51) \quad \begin{aligned} &\log KG_T(t)/G_{\bar{T}}(t) + COV \left(\int_t^T r_s ds, L \middle| \mathcal{A}_t \right) \\ &+ \frac{1}{2}VAR(L|\mathcal{A}_t) - COV \left(L, \int_t^T r_s ds \middle| \mathcal{A}_t \right) \\ &= \log KG_T(t)/G_{\bar{T}}(t) + \frac{1}{2}VAR(L|\mathcal{A}_t). \end{aligned}$$

Therefore, (9.50) becomes

$$(9.52) \quad z_2 = \frac{1}{\sqrt{VAR(L)}} \left(\log KG_T(t)/G_{\bar{T}}(t) + \frac{1}{2}VAR(L) \right)$$

and

$$(9.53) \quad E(\mathbf{1}_{Z > z_2}) = 1 - N(z_2) = N(-z_2) = N(d_2),$$

where

$$(9.54) \quad d_2 = \frac{1}{\sqrt{VAR(L)}} \left(\log G_{\bar{T}}(t)/KG_T(t) - \frac{1}{2}VAR(L) \right),$$

as specified in statement of the lemma. The formula for the cash-or-nothing put option is derived using call-put parity. \square

Theorem 4. *Let r_t be a process for the short rate which satisfies Condition 1. Then the formulae for the expectations f_5 and f_6 in (9.1) are given by*

$$(9.55) \quad \begin{aligned} f_5(t, T, K, \bar{T}) &= G_{\bar{T}}(t)N(d_1) - KG_T(t)N(d_2) \\ f_6(t, T, K, \bar{T}) &= -G_{\bar{T}}(t)(1 - N(d_1)) + KG_T(t)(1 - N(d_2)), \end{aligned}$$

where

$$(9.56) \quad \begin{aligned} d_1 &= \frac{1}{2}\sigma_G + \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} \\ d_2 &= -\frac{1}{2}\sigma_G + \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= VAR \left(E \left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T \right) \middle| \mathcal{A}_t \right). \end{aligned}$$

Proof. Expressing the call option as a combination of an asset-or-nothing call option and a cash-or-nothing call option, we have

$$(9.57) \quad f_5(t, T, K, \bar{T}) = f_1(t, T, K, \bar{T}) - Kf_3(t, T, K, \bar{T}).$$

Inserting (9.37) and (9.49) into (9.57) gives

$$(9.58) \quad \begin{aligned} c_{T,K,G_{\bar{T}}} &= G_{\bar{T}}(t)E(\mathbf{1}_{Z>z_1}) - KG_T(t)E(\mathbf{1}_{Z>z_2}) \\ &= G_{\bar{T}}(t)N(-z_1) - KG_T(t)N(-z_2) \end{aligned}$$

as required. The formula for the put option is derived from (9.58) and call-put parity. \square

When the short rate obeys a Vasicek process, the pricing formulae for call and put options on the ZCB $G_{\bar{T}}$ follow as a corollary of the above theorem.

Theorem 5. *Under the Vasicek model, for a strike price K and valuation time t , the price of a T -expiry call option on a \bar{T} -maturity ZCB is*

$$(9.59) \quad c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)N(h) - KG_T(t)N(h - \sigma_G)$$

and the price of a T -expiry put option on a \bar{T} -maturity ZCB is

$$(9.60) \quad p_{T,K,G_{\bar{T}}}(t) = -G_{\bar{T}}(t)N(-h) + KG_T(t)N(-h + \sigma_G),$$

where

$$(9.61) \quad h = \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} + \frac{1}{2}\sigma_G$$

$$(9.62) \quad \sigma_G = \sigma B(T, \bar{T}) \sqrt{\frac{1}{2\kappa}(1 - \exp(-2\kappa(T - t)))}$$

and

$$(9.63) \quad B(t, T) = \begin{cases} \frac{1}{\kappa}(1 - \exp(-\kappa(T - t))), & \text{if } \kappa > 0 \\ T - t, & \text{if } \kappa = 0 \end{cases}.$$

Proof. Using Theorem 4 we must compute

$$(9.64) \quad \sigma_G^2 = VAR\left(E\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right).$$

Firstly, from Lemma 2,

$$(9.65) \quad \int_T^{\bar{T}} r_s ds = r_T B(T, \bar{T}) + \bar{r}(\bar{T} - T - B(T, \bar{T})) + \sigma \int_T^{\bar{T}} B(u, \bar{T}) dZ_u$$

and therefore

$$(9.66) \quad E\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) = r_T B(T, \bar{T}) + \bar{r}(\bar{T} - T - B(T, \bar{T}))$$

from which we have

$$(9.67) \quad \sigma_G^2 = B(T, \bar{T})^2 VAR\left(r_T \middle| \mathcal{A}_t\right).$$

Secondly, from Lemma 1 we have

$$(9.68) \quad VAR(r_T | \mathcal{A}_t) = \sigma^2 \left(B(t, T) - \frac{1}{2} \kappa B(t, T)^2 \right),$$

which simplifies to

$$(9.69) \quad VAR(r_T | \mathcal{A}_t) = \sigma^2 \frac{1 - \exp(-2\kappa(T - t))}{2\kappa}.$$

It follows that

$$(9.70) \quad \sigma_G^2 = B(T, \bar{T})^2 \sigma^2 \frac{1 - \exp(-2\kappa(T - t))}{2\kappa}$$

and we arrive at the result. \square

Equation (9.59) agrees with the formula for the price of a call option on a zero coupon bond given in Jamshidian [1989]. However, Jamshidian has made an assumption of risk neutral dynamics and has calculated expectations involving log-normal ZCB prices, omitting many details of the proof, to arrive at the result, whereas we have calculated expectations under the real world measure, giving all details of the proof.

When the short rate obeys a Hull-White process, as in (3.4), the pricing formulae for call and put options on the \bar{T} -maturity ZCB follow as a corollary of Theorem 4.

Theorem 6. *Under the Hull-White model, for a strike price K and valuation time t , the price of a T -expiry call option on a \bar{T} -maturity ZCB is*

$$(9.71) \quad c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)N(h) - KG_T(t)N(h - \sigma_G)$$

and the price of a T -expiry put option on a \bar{T} -maturity ZCB is

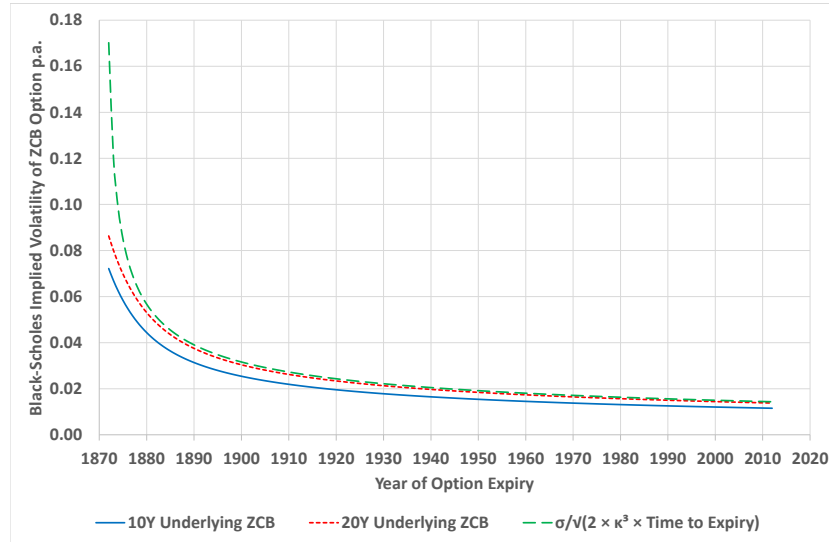
$$(9.72) \quad p_{T,K,G_{\bar{T}}}(t) = -G_{\bar{T}}(t)N(-h) + KG_T(t)N(-h + \sigma_G),$$

where

$$(9.73) \quad h = \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} + \frac{1}{2} \sigma_G$$

$$(9.74) \quad \sigma_G = B(T, \bar{T}) \sqrt{\int_t^T \exp\left\{-2 \int_u^T a(\tau) d\tau\right\} \sigma(u)^2 du}$$

FIGURE 6. Comparison of asymptotic formula with Black-Scholes implied option volatilities of options on 10Y and 20Y zero coupon bonds based at year 1871.



and

$$(9.75) \quad B(t, T) = \int_t^T \exp \left\{ - \int_t^s a(\tau) d\tau \right\} ds.$$

In Figure 6 the Black-Scholes implied volatilities of options on ten-year and twenty-year ZCBs are shown, based upon the parameters in (5.4). There is good agreement with the theoretical asymptotic formula of the Black-Scholes implied volatility, obtained as

$$(9.76) \quad \sigma_{BS} = \frac{\sigma_G}{\sqrt{T-t}} \rightarrow \frac{\bar{\sigma}}{\sqrt{2\bar{a}^3(T-t)}},$$

as $\bar{T} \rightarrow \infty$.

10. CONCLUSIONS

We have supplied actuarial pricing formulae for zero coupon bonds and call and put options on zero coupon bonds under the Vasicek model, which straightforwardly extend to other Gaussian short rate models. There is scope to generalise these formulae to multifactor Gaussian short rate models in subsequent work. Also, we have shown that extended Vasicek models give rise to sensible long-term bond yields when suitable conditions are imposed on the model parameters.

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