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Skitmore, Martin

Published in:
CME 25 Conference - Proceedings of the Inaugural Construction Management and Economics 'Past, Present and Future' conference

FIRST AND SECOND PRICE INDEPENDENT VALUES SEALED BID PROCUREMENT AUCTIONS: SOME SCALAR EQUILIBRIUM RESULTS

Martin Skitmore
Professor of Construction Management and Economics
School of Urban Development
Queensland University of Technology
Gardens Point
Brisbane Q4001
Australia

rm.skitmore@qut.edu.au

4 May 2007

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ABSTRACT

A great body of knowledge exists on the theory of auctions and competitive bidding that is of potential relevance to construction contract tendering. Most of this, however, contains assumptions – such as perfect information – that are unlikely to be tenable in practice. The aim, therefore, is to examine the effects of relaxing some of the more restrictive of these assumptions in the construction tendering context. In particular, the effects of additive and multiplicative (scalar) mark-ups in equilibrium are examined for first and second price auctions in situations where bidders have different, uncertain, costs. This is illustrated firstly by Monte Carlo simulation – by which bids generated randomly from a normal distribution for six bidders and mark-ups applied systematically for each bidder in turn until equilibrium is reached. An extensive numerical analysis is then applied to obtain equilibrium results for both mark-up values and expected profit from the simple symmetric case through to more complex asymmetric cases for the uniform and normal distributions.

In general, it is found that first price auction bidders with relatively high c_v levels and a larger number of bidders involved bid higher in equilibrium but can expect little profit unless the number of bidders involved is small. Where there are asymmetries, stronger bidders (i.e., those with lower costs and less variability) bid much higher and achieve much higher profits in equilibrium. From the seller's point of view, it is cheaper, in equilibrium, to have a homogeneous group of low variability bidders. The work contributes to the body of knowledge on the economic theory of auctions by closing some of the gap between theory and practice.

Keywords: Auction theory, tendering, bidding, equilibrium, procurement

INTRODUCTION

While the practice of auctioning goes back to ancient times[1], the earliest academic treatments are relatively recent, with the contributions of Friedman (1956) from an operations research (decision theoretic) perspective, Vickery (1961) from a game theoretic perspective and Gates (1967) from what has been termed the tendering theory perspective (Runeson and Skitmore 1999). In general, decision and tendering theory seek to inform bidders while game theory seeks to inform sellers. All three approaches have some impractical assumptions. Decision theory (DT), for example, is essentially static, in that it assumes any given bidder's opponents to bid with either a random or constant mark-up. Game theory (GT) on the other hand assumes all bidders somehow always bid optimally irrespective of the value of their cost estimates.

[1] Casady (1967) mentions a report by the Greek historian Herodotus, who described the sale by auction of women to be wives in Babylonia around the fifth century BC.

Of the three, progress has been dominated by the development of the game theoretic approach into a full-blown Bayesian-Nash equilibrium theory, now termed Auction Theory (AT), under the standard economic assumption of rational utility maximisation – so that now “the auction problem can be understood by applying the usual logic of marginal revenue versus marginal cost” (Klemperer, 1999: 312)[2]. One of the major outcomes of this theoretical development has been to discover the equilibrium bidding strategies for independent *private value* (IPV) auctions. This assumes an idealised form of valuation process by which, in procurement auction terms, each bidder estimates the costs involved perfectly accurately. For example, Vickrey (1961) showed that if bidders are symmetric, that is, the resulting bids are assumed to drawn from the same probability distribution, the expected payment for the client/building owner in English first-price (open-cry), sealed-bid, second-price sealed-bid (Vickrey) and Dutch (descending) auctions is the same in equilibrium.

As an alternative to IPV auctions, in which it is assumed that bidders have perfectly estimated but different true costs, the *common value* (CV) model has been studied, in which all bidders are assumed to have the same, but imperfectly estimated, true costs, (eg., Wilson 1969). Clearly, the private and common value assumptions are special cases of a more general model which contains *both* imperfect information *and* different costs for each bidder. One version of this that has received considerable attention (eg., Myerson, 1981; Riley and Samuelson, 1981; Milgrom and Weber, 1982) is based on the idea of signals. Here it is assumed that each bidder receives a private value signal (cost estimate), but allows each bidder’s value (cost) to be a function of all the signals (other bidders’ cost estimates). With a suitable definition of this function in terms of the assumed conditional probabilities involved, Milgrom and Weber (1982) were able to develop the general model needed, termed the *affiliated values* model, by using a natural generalisation of the monotone likelihood ratio property commonly used in statistical models. This provides several equilibrium results, the most important of which is that the English auction generates the lowest bids followed by the second-price and, finally, the Dutch and first-price auction.

Milgrom and Weber’s work, however, is concerned with the general properties of symmetric auction models when types (values/signals) are *not* independently distributed (Monteiro and Moreira 2006:1), making the affiliation assumption, as Milgrom and Weber point out, necessarily restrictive. Although, as they say, it may accord well with the qualitative features of some situations, such as the sale of works of art, there are many other situations where it does not (Monteiro and Moreira 2006:1; de Castro 2004). In fact, de Castro (2004) is particularly critical, claiming the affiliated values assumption to be “very restrictive”; much more cumbersome to manipulate theoretically, with the monoticity of equilibrium hard to maintain; and leading to conclusions that are misleading if applied to reality. In his view, a return to the search for non-monotonic equilibria is urgently needed, citing Araujo *et al*’s (2003) general existence result of non-monotonic symmetric equilibria with independent types. Araujo *et al* (2004), among others, have continued this work to examine multidimensional situations. Meanwhile Lebrun (1996, 1999) has obtained some results for asymmetric first price auctions, that is when bidders’ values are differently distributed, while Cantillon (2004) has considered both first and second price asymmetries. Guth *et al* (2004) provide a summary of much of the asymmetry work. No treatment appears yet to have been made of the equilibria for the asymmetric general independent

[2] The most notable contributions have come from Griesmer *et al* (1967), Wilson (1969, 1977), Milgrom (1979, 1981), Riley and Samuelson (1981), Myerson (1981) and Milgrom and Weber (1982) – see Klemperer (1999) for a comprehensive account.

values (GIV) model which contains both imperfect information *and* different costs – most likely because of the difficulties involved in finding analytical solutions (Rothkopf *et al* 2003: 72)

To examine the theory further for construction contract auctions, a starting point is to return to the original theme and consider the GIV model where bidders have, independently, both different costs and imperfect estimates of them. In addition, unlike AT where unbounded rationality is assumed, the goal is to maximise profits by the more practical means of mark-up manipulation. This involves finding equilibrium in DT-like scalar strategies rather than AT functions (Rothkopf *et al* 2003:73).

Equilibrium multiplicative mark-up strategies in a *symmetric* common value (imperfect estimates but same costs) sealed bid game theoretic setting have been reported in several studies. Rothkopf (1969, 1980a), for example, solves the *n* bidder Weibull distributed first price (FPA) situation analytically, while Oren and Rothkopf (1975), extend this to the situation where a bidder's strategy in one auction affects his competitors' behaviour in subsequent auctions, modelling bidding in a sequence of auctions as a multistage control process. Smith and Case (1975), on the other hand, consider the two bidder loglogistic common value FPA situation for both pure and mixed (randomised) strategies, while Rothkopf (1991) also considers the *n* bidder common value Weibull FPA and second price (SPA) situations in which bidders may submit two or more bids and then withdraw some bids after bids are opened.

For the *asymmetric* situation, Rothkopf (1969, 1980a) has solved the equilibrium multiplicative mark-up strategies for the two bidder *common variance* Weibull distributed FPA situation analytically, and the *n* bidder case numerically[3]. No results have reported for the fully asymmetric imperfect estimates case, where both location and scale parameters are unique to each bidder. Neither have any equilibrium results been reported for scalar strategies other than multiplicative for either the symmetric or asymmetric situation, with the exception of Rothkopf (1980b), who found, analytically, the equilibrium linear (affine) FPA mark-ups in the Weibull common value *n* bidder situation.

In general, therefore, it is concluded that, despite the difficulties involved in equilibrium asymmetric modelling with cost uncertainties, multiplicative mark-up models at least have had some success. In this paper, both the equilibrium multiplicative *and* additive mark-ups are considered within a general linear mark-up strategy. First, an example is provided describing how results may be obtained by straightforward Monte Carlo simulation to illustrate the simple concept underlying the analysis and identify some of the practical problems involved. Next, a numerical analysis is undertaken for the FPA and SPA for the uniform and normal composite densities and the results provided for some of the more obvious regularities detected. Finally, some practical observations are volunteered on the relevance of the analysis to construction contract bidding in practice.

[3] See Rothkopf *et al* (2003) for references to other examples of multiplicative mark-ups in the game theory approach.

SIMULATION METHOD

Analysis

Fig 1 shows the FPA results of simulating 100,000 values for each 0.1% change in (multiplicative) mark-up values by a bidder, assuming other bidders are all bidding with the same markup as each other. In this illustration, a typical construction contract auction situation is assumed in which there are 6 bidders in total, each drawing cost estimates from a normal distribution with 4.3% coefficient of variation. Overheads are calculated by the CIC formula converted to rebased HK$, with project size (HK$value) being loglognormal (mean=2.872829, sd=0.061078).

To understand Fig 1, first assume all opposing bidders apply a 12% mark-up. The expected profit is then recorded for the references bidder over series of reference bidder mark-up values. The results of this are shown in Fig 1 as the curve marked “12%”. The optimal mark-up, i.e., the mark-up value that provides the maximum expected profit, occurs at the upper turning point of this curve at 6.7% mark-up (point A), giving an expected profit of $21.4m.

Now assume all opposing bidders apply a 11% mark-up. Again the reference bidder’s expected is recorded over a series of reference bidder mark-up values and plotted, this time at the curve marked 11%. In this case the optimal mark-up is 6.5%, giving an expected profit of $15.5m. Fig 1 shows the results of this repeated for 10%, … , 5%. Line A-B connects the optimal mark-up values.

For the game theoretic solution the usual approach is to use the Nash criterion. That is, the solution occurs where any alternative solution for any player produces a worst result overall. This can be reached by trial and error as follows. In this case, from Fig 1, assume that all bidders bid at 12% mark-up. Now, as mentioned above, the reference bidder can maximise his profits by bidding a 6.7% mark-up. Unlike the DT approach, where competitors are assumed to be unable to change, the game theoretic approach assumes that all bidders know that bidding at 6.7% mark-up is best against competitors bidding at 12% mark-up. In other words, it has to be allowed that all bidders will bid at 6.7% mark-up under the assumption that their competitors are bidding at 12 % mark-up.

Now, assuming that all bidders are bidding at 6.7% mark-up. This will be on the line C-D. Our bidder’s best mark-up now is the point on A-B at the same value of the y-axis, ie., around 8.25% mark-up. Following the same reasoning then, if all bidders bid at 8.25% mark-up, Our bidder should bid at around 7% mark-up. Repeating this process enough times results in convergence at 7.4% mark-up, where no bid higher or lower by any bidder will produce a better result. This, then, is the equilibrium solution and occurs ***where the two lines A-B and C-D intersect***, in this case producing an expected profit of approximately $2m.

Figs 2 and 3 shows the solution of the SPA. Again A-B gives the optimal results, showing 0% mark-up to be optimal when competitors all bid at greater than 5% mark-up, but increasing quite rapidly when they bid less. The Nash outcome (Fig 3) is 3.6%, producing an expected profit of $1.6m.

Comment

A few issues arise out of this illustrative analysis by simulation. One is that the expected profit for some optimal mark-up values is sometimes negative. That is, it can be an expected loss. Of course, this is unlikely to be the case in a live situation as presumably bidders will prefer not to bid at all rather than make a loss under equilibrium[4].

Another issue is that using Monte Carlo simulation, although very fast on a modern PC, is still quite time consuming (Fig 1 takes around one hour to produce). While it is useful enough for a one-off auction situation with a fixed set of parameters, it is unlikely to be practicable in an extensive analysis aimed at identifying general relationships and regularities. To do this necessarily requires a massive reduction in computing time. In previous similar studies, the most popular approach to this is to derive general results by analytical means. However, as has been already observed, this is seldom possible beyond the relatively simple symmetric IPV and CV assumptions.

An intermediate approach that has been used previously with some success for more complex situations is numerical analysis. This involves deriving formulae that can be solved by a means other than by finding the solution of a set of differential equations. In this case, the formulae for calculating the expected profit is used together with numerical maximisation software from the Numerical Algorithms Group (NAG) Library. The formulae developed are presented in the Appendix for the general linear (additive and multiplicative) mark-up case.

NUMERICAL METHOD: SOME FPA RESULTS

The Nash solution is easily found numerically by using **(A.2)** to first find the mark-up values, $\tilde{v}_{1i}$ and $\tilde{v}_{2i}$ that maximise expected profit $\tilde{E}_i = E_{(k)i}[\theta]$ for bidder i, then finding $\tilde{v}_{1j}$ and $\tilde{v}_{2j}$ that maximises $\tilde{E}_j = E_{(k)j}[\theta]$ for the next bidder $(j = 1,\ldots,n;\, j \neq i)$assuming the first bidder is using $\tilde{v}_{1j}$ and $\tilde{v}_{2j}$ etc, after assuming appropriate starting values. Upon convergence, $v_{1i}^* = \tilde{v}_{1i}$ and$v_{2i}^* = \tilde{v}_{2i}$, etc $(i = 1,\ldots,n)$are the equilibrium mark-up values, with associated equilibrium expected profits,$E_i^* = \tilde{E}_i$, etc.

Additive equilibrium mark-up (v_1^*)

In reality, it is most likely that the true parameters for all bidders will be different, ie., $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$ (Skitmore, 1991). However, it is of interest to see what the effects are of some simplifying assumptions. The simplest of these is where the value of all bidders' parameters are equal to our own, ie., $\mu_1 = \mu_2 = \ldots = \mu_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_n$. This is equivalent to the symmetric *common value* model, many of the properties of which have already been established for the formless bid function. However, these are now considered here for the

[4] I am indebted to Michael Rothkopf for pointing this out.

first time for the additive equilibrium mark-up, v_1^*, by fixing $v_2 = 1$. Then the effects are considered of $\mu_1 = \mu_2 = \ldots = \mu_n$ while retaining $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$. This is followed by the alternative $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ with $\sigma_1 = \sigma_2 = \ldots = \sigma_n$. Finally, the most general case is examined of $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$.

Common parameters (the symmetrical common value model)

Fig 4 gives the results for v_1^* and E^* for the 2 bidder symmetric common value model for the uniform distribution for a range of σ values. This indicates that $v_1^* = \sigma\sqrt{3}$ (or $v_i^* = \frac{b-a}{2}$ where b and a are the upper and lower supports). *This is the dominant strategy and true for any value of n and μ.* Also, $E^* = \frac{2\sigma\sqrt{3}}{n(n+1)}$ (or $E^* = \frac{b-a}{n(n+1)}$), which is true for any value of μ. The equilibrium expected revenue (client's payment) is $R^* = \mu + nE^*$ (or $R^* = \frac{a(n-1)+b(2b+n)}{2(n+1)}$).

v_1^* and E^* are also independent of μ (but not n) for the normal distribution provided σ is proportional to μ. This enables the results, although additive, to be stated in more conventional terms as a percentage over a range of coefficients of variation (c_v). These are shown in Fig 5 and Fig 7 for v_1^* and E^* respectively for both the normal (blue) and uniform (red) distributions and are true for any value of μ. As these Figs show, the relationship between c_v and v_1^* (and c_v and E^*) are linear with both $v_1^* \to 0$ and $E^* \to 0$ as $c_v \to 0$. Fig 6 gives an alternative representation of the v_1^* results for the normal distribution, showing an abnormality of values for smaller values of n as c_v increases.

Common μ (the asymmetric common value model)

For the asymmetric common value model, $\mu_1 = \mu_2 = \ldots = \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$. Let $\sigma_{(1)}$ denote the smallest value of σ_i $(i = 1,2,\ldots,n)$ then, for the uniform distribution, $v_{1i}^* = \sigma_i\sqrt{3}$ for $\sigma_i > \sigma_{(1)}$ as with the symmetric model, while $v_{1i}^* > \sigma_i\sqrt{3}$ for $\sigma_i = \sigma_{(1)}$. *This is again the dominant strategy and true for any value of n and μ.* Also, $E_i^* < \frac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i > \sigma_{(1)}$ and $E_i^* > \frac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i = \sigma_{(1)}$ - again true for any value of μ ($R^* = \mu + \sum_i^n E_i^*$ for all asymmetric cases).

For the normal distribution, the only discernable trend is for the 2 bidder case, where v_1^* and E^* are proportional to $\frac{\sigma_i}{\sigma_j}$. That is $v^*_{1\frac{\sigma_i}{\sigma_j}} = \sigma_i v^*_{1\frac{1}{\sigma_j}}$ and $E^*_{\frac{\sigma_i}{\sigma_j}} = \sigma_i E^*_{\frac{1}{\sigma_j}}$ Fig 8 shows the results for $1/\sigma_j$ (v_1^* and E^* are the same for both bidders) for both uniform and normal distributions. These are true for any value of μ.

Common σ

Similarly, where $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_n$, $v_{1i}^* = \sigma_i \sqrt{3}$ for the uniform distribution except for the bidder with the lowest μ value, in which case $v_1^* > \sigma_i \sqrt{3}$ for that bidder. $E_i^* < \frac{2\sigma_i \sqrt{3}}{n(n+1)}$ except for the bidder with the lowest μ value, in which case $E_i^* > \frac{2\sigma_i \sqrt{3}}{n(n+1)}$ for that bidder.

For the normal distribution, the only discernable trend again is for the 2 bidder case. Figs 9 and 10 show the values of v_{1i}^* and E_i^* for a range of $\mu_i - \mu_j$ for values of σ common to each bidder for both uniform and normal distributions.

Heterogeneous case

For the complex case of $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$, no general results are obtainable except for the 2 bidder common c_v case. Figs 11 and 12 give the results for v_{1i}^* and E_i^* respectively for c_v =0.01, 0.05 and 0.10 for a range of $\frac{\sigma_i}{\sigma_j}$ values for both uniform and normal distributions for $\mu_i = 1$ (the results being a multiple of μ_i, i.e., $v_{1i\mu_i}^* = \mu_i v_{1i}^*$ and $E_{i\mu_i}^* = \mu_i E_i^*$).

Multiplicative equilibrium mark-up (v_2^*)

Common parameters (the symmetrical common value model)

For the symmetric common value model, v_2^* and E^* are proportional to the coefficient of variation, c_v. Fig 13 gives the results for the 2 bidder situation for the uniform and normal distributions. This shows $v_2^* \to 1$ as $c_v \to 0$ and $v_2^* \to 0$ as $c_v \to \infty$. In addition, for the uniform distribution, $v_2^* = \frac{1}{2} + \frac{\sqrt{3}}{6}$ at $c_v = 1$ and $v_2^* = 1 + \frac{\sqrt{3}}{3}$ at $c_v = \frac{1}{2}$, reaching a maximum of exactly 1.6

where c_v is exactly 0.625. For the normal distribution, the $v_2^* = 1.3745$ at $c_v = 1$ and $v_2^* = 2.0459$ at $c_v = \frac{1}{2}$, reaching a maximum of 2.1 where c_v is 0.59.

The associated expected profit is a multiple of σ, i.e., $E_\sigma^* = \sigma E^*$ for both uniform and normal distributions. Fig 13 shows the values of E^* for $\sigma = 1$. This indicates that $E^* \to \frac{\sigma}{\sqrt{3}}$ as $c_v \to 0$ and $E^* \to 0$ as $c_v \to \infty$. In addition, for the uniform distribution, $E^* = -\frac{\sigma}{3}$ at $c_v = 1$, reaching a minimum of exactly $-\frac{\sigma\sqrt{3}}{5}$ where c_v is exactly $\frac{\sqrt{3}}{2}$. For the normal distribution, E^* reaches a maximum of 0.6437 where c_v is 0.202 and a minimum of -0.3057 at c_v=1.408.

Fig 14 shows the results for the *n* bidder situation up to $c_v \le 0.3$ (convergence becomes highly dependent on starting values for *n*>2 at c_v>0.3). This indicates that $v_2^* \to 1$ for any *n* bidders as $c_v \to 0$. The associated expected profit for the *n* bidder situation is again a multiple of σ and is shown in Fig 15 for $\sigma = 1$.

Asymmetric model (common c_v)

For the case of 2 bidders with a common c_v, the v_2^* is proportional to their standard deviations. Fig 16 shows the v_2^* for bidder *i* over a range of $\frac{\sigma_i}{\sigma_j}$ values where both bidders have typical values of c_v=0.01, 0.05 and 0.10. This suggests that $v_2^* \to \infty$ as $\frac{\sigma_i}{\sigma_j} \to 0$ and $v_2^* \to 1 + 2c_v$ as $\frac{\sigma_i}{\sigma_j} \to \infty$ (the results for $\frac{\sigma_i}{\sigma_j} = 1$ are given in the previous section). Although convergence does occur for typical values, no easily discernable patterns of results were found for expected profit.

NUMERICAL METHOD: SOME SPA RESULTS

Additive equilibrium mark-up (v_1^*)

For the uniform distribution, no second price v_1^* can be found except for the symmetric common value model with *n*=2, where the v_1^* is zero (if and only if the starting point is zero), with expected profit of $E^{*} = \frac{\sigma_i\sqrt{3}}{n(n+1)} = \frac{\sigma_i\sqrt{3}}{6}$ (exactly half the value of the FPA[5]) and expected revenue (client's payment) again of $R^{*} = \mu + nE^{*} = \mu + 2E^{*}$.

Second price solutions for the normal distribution are only obtainable for the common value model and when the starting point is zero. As with the first price arrangement, the v_1^* and associated expected profit values are a multiple of σ. Table 1 shows these for $\sigma = 1$. With the exception of R^*, which is again $\mu + nE^*$, all results are independent of the value of μ. It is interesting to note that expected profit at the SPA v_1^* is close to half the expected profit at the FPA v_1^* – equivalent to the known affiliation result that the SPA generates a better return to the seller than does the first price equivalent (Milgrom and Weber, 1982:1095).

	1st price				**2nd price**			
n	v_1^*	$P_{(1)}(v_1^*, v_2 = 1)$	E^*	R^*	v_1^*	$P_{(1)}(v_1^*, v_2 = 1)$	E^*	R^*
2	1.7725	0.5000	0.6041	1.2083	0.0000	0.5000	0.2821	0.5642
3	1.5074	0.3333	0.2204	0.6611	0.3257	0.3333	0.1086	0.3257
4	1.5071	0.2500	0.1194	0.4777	0.5356	0.2500	0.0596	0.2386
5	1.5478	0.2000	0.0770	0.3848	0.6879	0.2000	0.0386	0.1929
6	1.5954	0.1667	0.0547	0.3282	0.8063	0.1667	0.0274	0.1645
7	1.6420	0.1429	0.0414	0.2898	0.9025	0.1429	0.0207	0.1451
8	1.6855	0.1250	0.0327	0.2619	0.9831	0.1250	0.0164	0.1309
9	1.7257	0.1111	0.0267	0.2406	1.0523	0.1111	0.0133	0.1200
10	1.7625	0.1000	0.0224	0.2238	1.1127	0.1000	0.0113	0.1113
15	1.9093	0.0667	0.0116	0.1734				
30	2.1631	0.0333	0.0040	0.1235				

Table 1: Equilibrium results for the Standard Normal Distribution $(E[C] = \mu = 0)$

Multiplicative equilibrium mark-up (v_2^*)

Doesn't converge!

DISCUSSION

Being the first time such an analysis has been undertaken, perhaps the first and most important result of work detailed here is that there any results at all! Unlike the "higher" analysis (Rothkopf and Harstad 1994) for conventional analytical approach in AT, being reduced to obtaining numerical solutions is an uncertain business. First there is the problem of finding suitable starting values. Using standard minimising software routines, as has been done here, can make this a sensitive issue as was often the case for the $n > 2$, and for all the SPA analyses attempted. Next there is the problem of suboptimal solutions and failure to converge. In several cases the iteration

[5] Note that, while AT predicts the FPA and SPA to be the same for this symmetrical case, this analysis, unlike AT, is restricted to the profit maximisation by mark-up manipulation, which may account for the difference in outcome.

loops around a ‘strange attractor’, fluctuating between two or three values for each bidder. Perhaps most important of all is the possibility of errors in the computer program. There is a profound lack of earlier analyses results upon which to carry out replication tests. In the end, it was possible to adapt the program to replicate some of Maskin and Riley (2000) and Li and Riley’s (1999) asymmetric IPV results although only in a limited way as they use bid functions rather than the scalar strategies used here. As a further precaution, the integration routines were all subject to intensive testing by Monte Carlo simulation.

Turning to the results themselves (Table 2 provides a summary), it is clear that a good number of ‘solutions’ are obtainable for the FPA symmetric and n=2 situations, with the uniform distribution in particular providing some elegant-looking relations that can surely be derived analytically. This is not surprising as the usual analytical approach is to start with the 2 bidder symmetric uniform situation before moving on to the n bidder situation, followed by a distribution free treatment (the asymmetric situation is often intractable).

Of course, for generality, the results have been obtained for a wide a range of parameters as possible. Construction contract bidding, however, is concerned with a more restricted range. The coefficient of variation, for example, is generally taken to be around 0.05 to 0.10 for construction bidding (Skitmore 1989: Table 7.2). Within this narrower range, the symmetric FPA results suggest the c_v to be rather more important than the number of bidders, with v_1^* ranging between 7-11% for 2-30 bidders at $c_v = 0.05$ to 15-22% for 2-30 bidders at $c_v = 0.10$ (Figs 5 and 6) for the, with a similar order of magnitude for v_2^* (Fig 14). The opposite effect occurs with the associated expected profit, however, where E^* is always small for a large number of bidders, irrespective of c_v. For a small number of bidders, E^* becomes significantly higher generally irrespective of c_v for the multiplicative mark-up (Fig 15) but only for larger c_v values for the additive mark-up (Fig 7) In terms of the differences between the uniform and normal assumption, the v_1^* for the uniform with any number of bidders is approximately the same as that for around eight normal bidders, and quite similar to the normal E^* (Fig 7). The v_2^* for the uniform on the other hand, is generally quite similar when a higher number of bidders is involved, which is the reverse of the situation for the normal distribution (Fig 14)

It is not possible to judge the effect of the number bidders in the asymmetric situation as no trends were observed for $n > 2$. For the 2 bidder situation though, the advantage of one (stronger) bidder having a smaller mean or variance is very much pronounced in the v_1^* and E^* values (Figs 9-12), with the results for the uniform and normal being very similar for common variance (Figs 9-12) but with the uniform v_1^* and E^* always being higher than the normal equivalent for the common mean (Fig 8). Similarly, the v_2^* results for the stronger (lower variance) bidder are also very striking (Fig 16), with the c_v level being much more important for the higher variance bidder.

CONCLUSIONS

This paper has presented some numerical equilibrium additive and multiplicative mark-up results for the uniform and normal 1st and 2nd price construction contract auctions, in which bidders are assumed to have both different costs and imperfect estimates of them. Overall, and bearing in mind the limitations mentioned below, it is clear from this analysis that FPA bidders with relatively high c_v levels and a larger number of bidders involved bid higher in equilibrium but can expect little profit unless the number of bidders involved is small. Where there are asymmetries, stronger bidders (i.e., those with lower costs and less variability) bid much higher and achieve much higher profits in equilibrium. From the seller's point of view, the implications are equally clear. It is cheaper, in equilibrium, to have a homogeneous group of low variability bidders – a result anticipated by Flanagan and Norman (1985), which they interpret as implying the need for good quality of information to bidders experienced in the type and size of contract under auction (p.159).

There are also indications that the use of SPA and greater numbers of bidders may also be beneficial to sellers, but the analysis is quite limited in this respect – particular as the cost of bidding has not been included. Other well known limitations of equilibrium analysis are worthy of mention. A major issue is that bidders might not behave optimally (Flanagan and Norman 1984: 155-6), which raises the question of what do about this (Thaler 1988) to at least take advantage of the opportunity costs at stake. As must be expected, the economic basis of equilibrium results assumes all bidders to be acting rationally and so such an analysis is certainly going to lie outside the boundary of economics. A further important issue concerns the limitations of auction theory as a branch of game theory, in that the competitor behaviour is assumed to be limited only to Nash type responses to the optimal moves of others. As Rothkopf and Harstad (1994) comment, many of the common assumptions of game theoretic models – symmetry, common knowledge, isolation, fixed number of bidders and unbending rules - are suspect from an applications point of view. Similarly, there is no allowance here for changes in market conditions (supply of, and demand for, contractors' services) or the capacity of the contractors (Runeson and Skitmore 1999). Finally, as has been mentioned in several seminal contributions in the construction literature (eg., Raftery, 1991; Hillebrandt, 2000; Runeson, 2000), the extent of the uncertainties involved in forecasting future costs as well as the behaviour of competitors and the market in general requires contract bidders to devote a far greater amount of energy and resources to marketing than is currently admitted in the economic theory of auctions.

ACKNOWLEDGEMENTS

This work has been continuing for many years, during which time I have been fortunate enough to have been ably assisted by a variety of people, the most prominent of which include mathematician/statisticians Roy Thomas, Ernest Wilde, Mike Patefield and John Pemberton all of Salford University. More recently, I am indebted to Ross McVinish of QUT and Flavio Menezes of the University of Queensland. The former, for his help in articulating the GIV model and its limitations. The latter, for summarising the main AT results and *cul de sacs* to date and the likely prospects for asymmetric models and mark-up form bid functions in general. Thanks also go to Michael Rothkopf for his comments on an earlier draft, drawing my attention to his work on equilibrium linear mark-ups and kind encouragement in general.

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APPENDIX: FORMULAE FOR NUMERICAL ANALYSIS

Let bidders for an auction be indexed $i = 1, \ldots, n$ and assume each makes a bid

$$t_i = v_{1i} + v_{2i} s_i \tag{A.1}$$

where the cost estimate s_i is a realisation of a random variable S_i with composite density $F(\lambda + \mu_i, \sigma_i)$ and v_{1i} and v_{2i} are the additive and multiplicative mark-ups respectively, λ is a parameter denoting the unique common value (nuisance) component of the auction (Skitmore, 1991; Hong and Shum 2003), and μ and σ are the mean and variance unique to each bidder. Suppose each bidder knows only the values μ_j, σ_j, v_{1j} and v_{2j} for all $j = 1, \ldots, n$ bidders. Let θ denote the vector of all these values μ_j, σ_j, v_{1j} and v_{2j}. Now, the *expected profit* for bidder *i* is

$$E_{(k)i}[\theta] = P_{(1)i}(\theta)\left(E\left[T_{(k)} \middle| i \text{ wins}\right] - E\left[C_i \middle| i \text{ wins}\right]\right) \tag{A.2}$$

where

$$E\left[T_{(k)} \middle| i \text{ wins}\right] = \int_{-\infty}^{\infty} (v_{1i} + v_{2i}x) \frac{g_{(k)i}(x;\theta)}{P_{(k)i}(\theta)} dx \tag{A.3}$$

with

$$P_{(k)i}(\theta) = \int_{-\infty}^{\infty} g_{(k)i}(x;\theta) dx \tag{A.4}$$

where (k) denotes the *k*th lowest bidder. Assuming estimates are unconditionally unbiased (Flanagan and Norman 1985), the expected cost given *i* wins, $E\left[C_i \middle| i \text{ wins}\right] = \mu_i$.

The function $g_{(k)i}(x;\theta)$ depends on the composite density assumed. For the Uniform density, where $a_i = v_{1i} + v_{2i}\left(\mu_i - \sigma_i\sqrt{3}\right)$ and $b_i = v_{1i} + v_{2i}\left(\mu_i + \sigma_i\sqrt{3}\right)$, it can be shown that

$$g_{(1)i}(x;\theta) = \frac{1}{b_i - a_i} \prod_{j \neq i}^{n} \frac{b_j - x}{b_j - a_j} \qquad (b \geq x \geq a) \tag{A.5}$$

and

$$g_{(2)i}(x;\theta) = \sum_{j_2 \neq i}^{n} \frac{1}{b_{j_2} - a_{j_2}} \frac{x - a_i}{b_i - a_i} \prod_{\substack{j_1 \neq i \\ j_1 \neq j_2}}^{n} \frac{b_{j_1} - x}{b_{j_1} - a_{j_1}} \qquad (b \geq x \geq a) \tag{A.6}$$

for the FPA and SPA respectively.

Likewise, for the Normal density, letting $\mu_i^{'} = v_{1i} + v_{2i}\mu_i$ and $\sigma_i^{'} = v_{2i}\sigma_i$,

$$g_{(1)i}(x;\theta) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \prod_{j \neq i}^{n} \int_{\frac{\sigma_i^{'}x + \mu_i^{'} - \mu_j^{'}}{\sigma_j^{'}}}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \qquad (\infty > x > -\infty) \tag{A.7}$$

and

$$g_{(2)i}(x) = \sum_{\substack{j_2=1 \\ j_2 \neq i}}^{n} \frac{e^{-\left(\frac{x-\mu'_{j_2}}{\sigma'_{j_2}\sqrt{2}}\right)^2}}{\sigma'_{j_2}\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu'_i}{\sigma'_i}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \prod_{\substack{j_1=1 \\ j_1 \neq i \\ j_1 \neq j_2}}^{n} \int_{\frac{x-\mu'_{j_1}}{\sigma'_{j_1}}}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \qquad \textbf{(A.8)}$$

for the FPA and SPA respectively.

The optimal value of the mark-ups, v_{1i} and v_{2i}, are then obtained numerically by finding the values that maximise **(A.2)** for bidder *i*[6]. Repeating this procedure in turn for each bidder until convergence then enables the equilibrium mark-ups to be obtained.

[6] Note that s_i, although known, is not necessary for this calculation.

Assumption	Additive mark-up		Multiplicative mark-up	
	Uniform	**Normal**	**Uniform**	**Normal**
Symmetric: $\mu_1 = \mu_2 = \ldots = \mu_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_n$	$v_1^* = \sigma\sqrt{3}$ for $n \geq 2$ $E^* = \dfrac{2\sigma\sqrt{3}}{n(n+1)}$	v_1^* and E^* are independent of μ and linear in c_v $v_1^* \to 0$ as $c_v \to 0$ $E^* \to 0$ as $c_v \to 0$	$v_2^* \propto c_v$ and $E^* \propto c_v$. $v_2^* \to 1$ as $c_v \to 0$ for $n \geq 2$ $v_2^* \to 0$ as $c_v \to \infty$ for n=2 $v_2^* = \dfrac{1+3^{-\frac{1}{2}}}{2}$ at $c_v = 1$ for n=2 $v_2^* = 1+3^{-\frac{1}{2}}$ at $c_v = \dfrac{1}{2}$ for n=2 $v_{2(\max)}^* = 1.6$ at $c_v = 0.625$ for n=2 $E_\sigma^* = \sigma E^*$ for $n \geq 2$ $E^* \to \dfrac{\sigma}{\sqrt{3}}$ as $c_v \to 0$ for n=2 $E^* \to 0$ as $c_v \to \infty$ for n=2 $E^* = -\dfrac{\sigma}{3}$ at $c_v = 1$ for n=2 $E_{(\max)}^* = -\dfrac{\sigma\sqrt{3}}{5}$ at $c_v = \dfrac{\sqrt{3}}{2}$ for n=2	$v_2^* \propto c_v$ and $E^* \propto c_v$. $v_2^* \to 1$ as $c_v \to 0$ for $n \geq 2$ $v_2^* \to 0$ as $c_v \to \infty$ for n=2 $v_2^* = 1.3745$ at $c_v = 1$ for n=2 $v_2^* = 2.0459$ at $c_v = \dfrac{1}{2}$ for n=2 $v_{2(\max)}^* = 2.1$ at $c_v = 0.59$ for n=2 $E_\sigma^* = \sigma E^*$ for $n \geq 2$ $E^* \to \dfrac{\sigma}{\sqrt{3}}$ as $c_v \to 0$ for n=2 $E^* \to 0$ as $c_v \to \infty$ for n=2 $E_{(\max)}^* = 0.6437$ at $c_v = 0.202$ for n=2 $E_{(\min)}^* = -0.3057$ at $c_v = 1.408$ for n=2
Asymmetric (common mean): $\mu_1 = \mu_2 = \ldots = \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$	$v_{1i}^* = \sigma_i\sqrt{3}$ for $\sigma_i > \sigma_{(1)}$ $E_i^* < \dfrac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i > \sigma_{(1)}$ $v_{1i}^* > \sigma_i\sqrt{3}$ for $\sigma_i = \sigma_{(1)}$ $E_i^* > \dfrac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i = \sigma_{(1)}$.	$v^*_{1\frac{\sigma_i}{\sigma_j}} = \sigma_i v^*_{1\frac{1}{\sigma_j}}$ for n=2 $E^*_{\frac{\sigma_i}{\sigma_j}} = \sigma_i E^*_{\frac{1}{\sigma_j}}$ for n=2	-	-

Asymmetric (common variance): $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_n$	$v_{1i}^* = \sigma_i\sqrt{3}$ and $E_i^* < \dfrac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i > \sigma_{(1)}$. $v_{1i}^* > \sigma_i\sqrt{3}$ and $E_i^* > \dfrac{2\sigma_i\sqrt{3}}{n(n+1)}$ for $\sigma_i = \sigma_{(1)}$.	v_{1i}^* and E_i^* depend on $\mu_i - \mu_j$ and σ for n=2	-	-
General asymmetric: $\mu_1 \neq \mu_2 \neq \ldots \neq \mu_n$ and $\sigma_1 \neq \sigma_2 \neq \ldots \neq \sigma_n$	v_{1i}^* and E_i^* depend on $\dfrac{\sigma_i}{\sigma_j}$ for common c_v for n=2 $v_{1i\mu_i}^* = \mu_i v_{1i}^*$ for n=2 $E_{i\mu_i}^* = \mu_i E_i^*$ for n=2	v_{1i}^* and E_i^* depend on $\dfrac{\sigma_i}{\sigma_j}$ for common c_v for n=2 $v_{1i\mu_i}^* = \mu_i v_{1i}^*$ for n=2 $E_{i\mu_i}^* = \mu_i E_i^*$ for n=2	v_{1i}^* depends on $\dfrac{\sigma_i}{\sigma_j}$ for common c_v for n=2 $v_2^* \to \infty$ as $\dfrac{\sigma_i}{\sigma_j} \to 0$ $v_2^* \to 1 + 2c_v$ as $\dfrac{\sigma_i}{\sigma_j} \to \infty$	v_{1i}^* depends on $\dfrac{\sigma_i}{\sigma_j}$ for common c_v for n=2 $v_2^* \to \infty$ as $\dfrac{\sigma_i}{\sigma_j} \to 0$ $v_2^* \to 1 + 2c_v$ as $\dfrac{\sigma_i}{\sigma_j} \to \infty$

Note: $R^* = \mu + \sum_i^n E_i^*$ throughout

Table 2: Summary of main equilibrium results

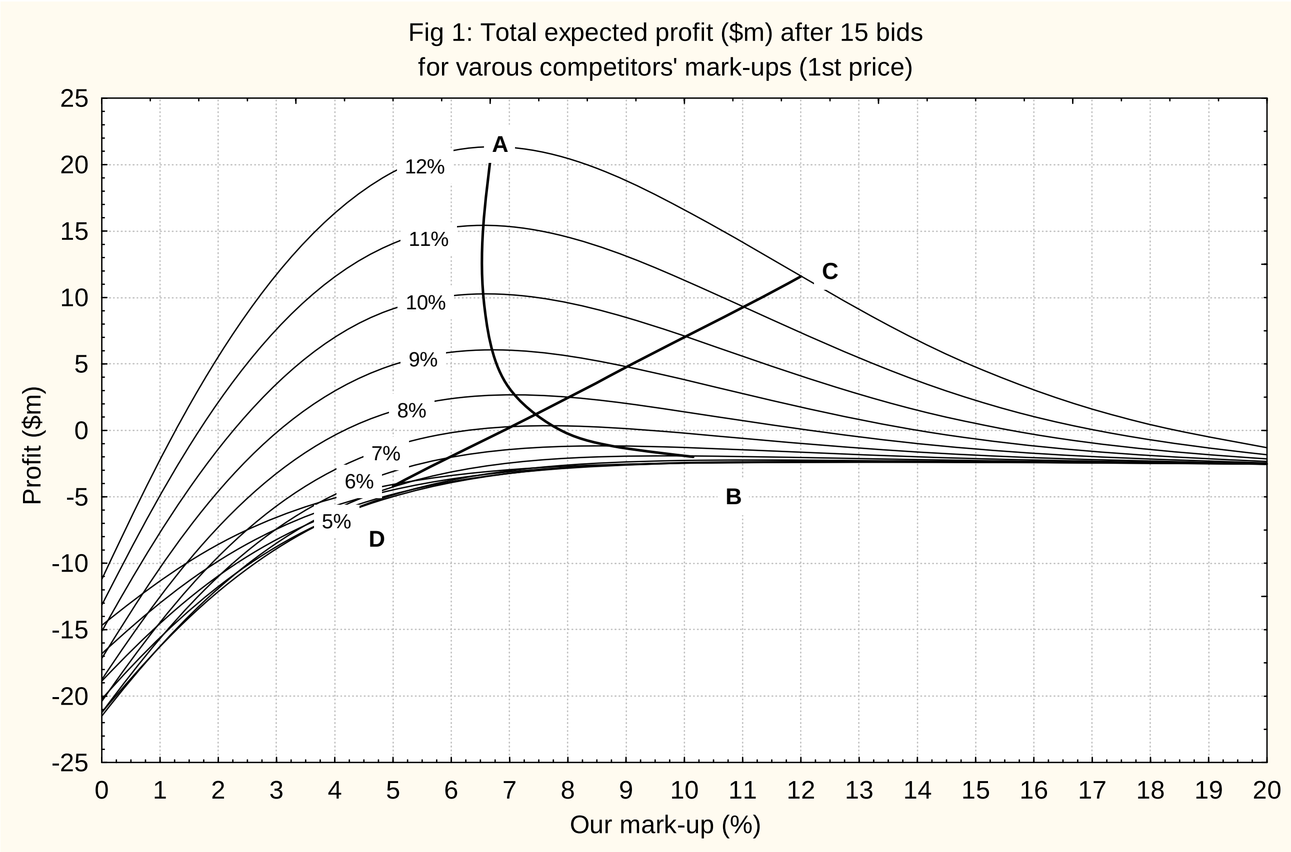
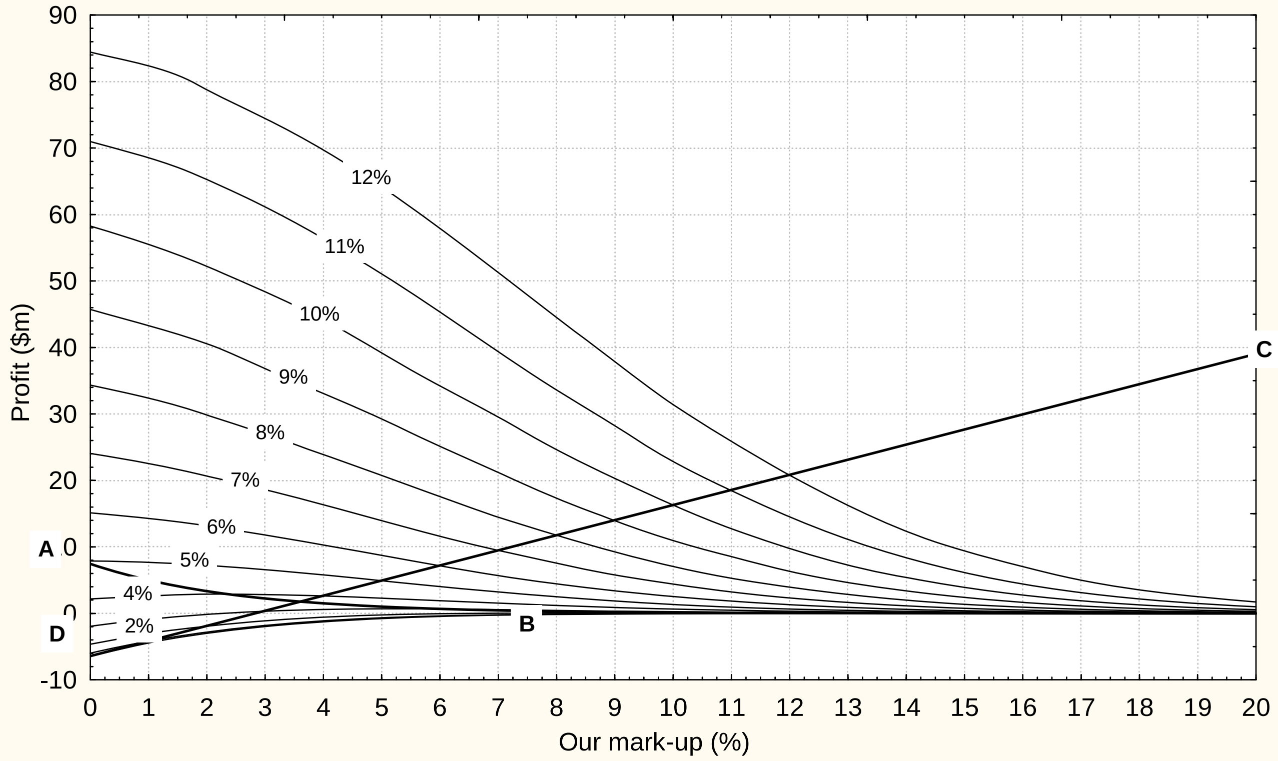


Fig 1: Total expected profit ($m) after 15 bids
for varous competitors' mark-ups (1st price)
Profit ($m)
Our mark-up (%)
12%
11%
10%
9%
8%
7%
6%
5%
A
B
C
D
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

Fig 2: Total expected profit ($m) after 15 bids
for varous competitors' mark-ups (2nd price)



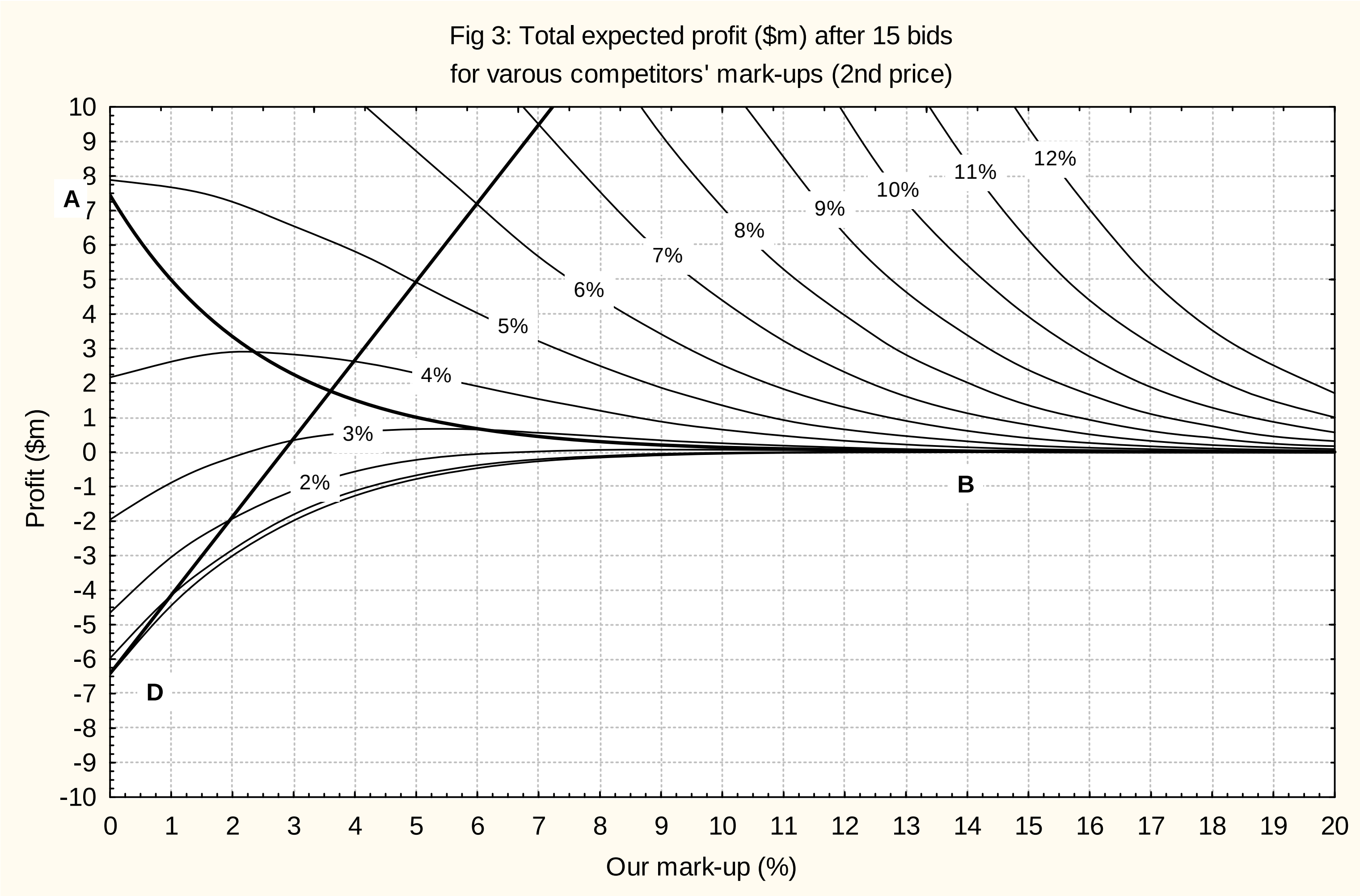


Fig 3: Total expected profit ($m) after 15 bids
for varous competitors' mark-ups (2nd price)
Profit ($m)
Our mark-up (%)
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
2%
3%
4%
5%
6%
7%
8%
9%
10%
11%
12%
A
B
D

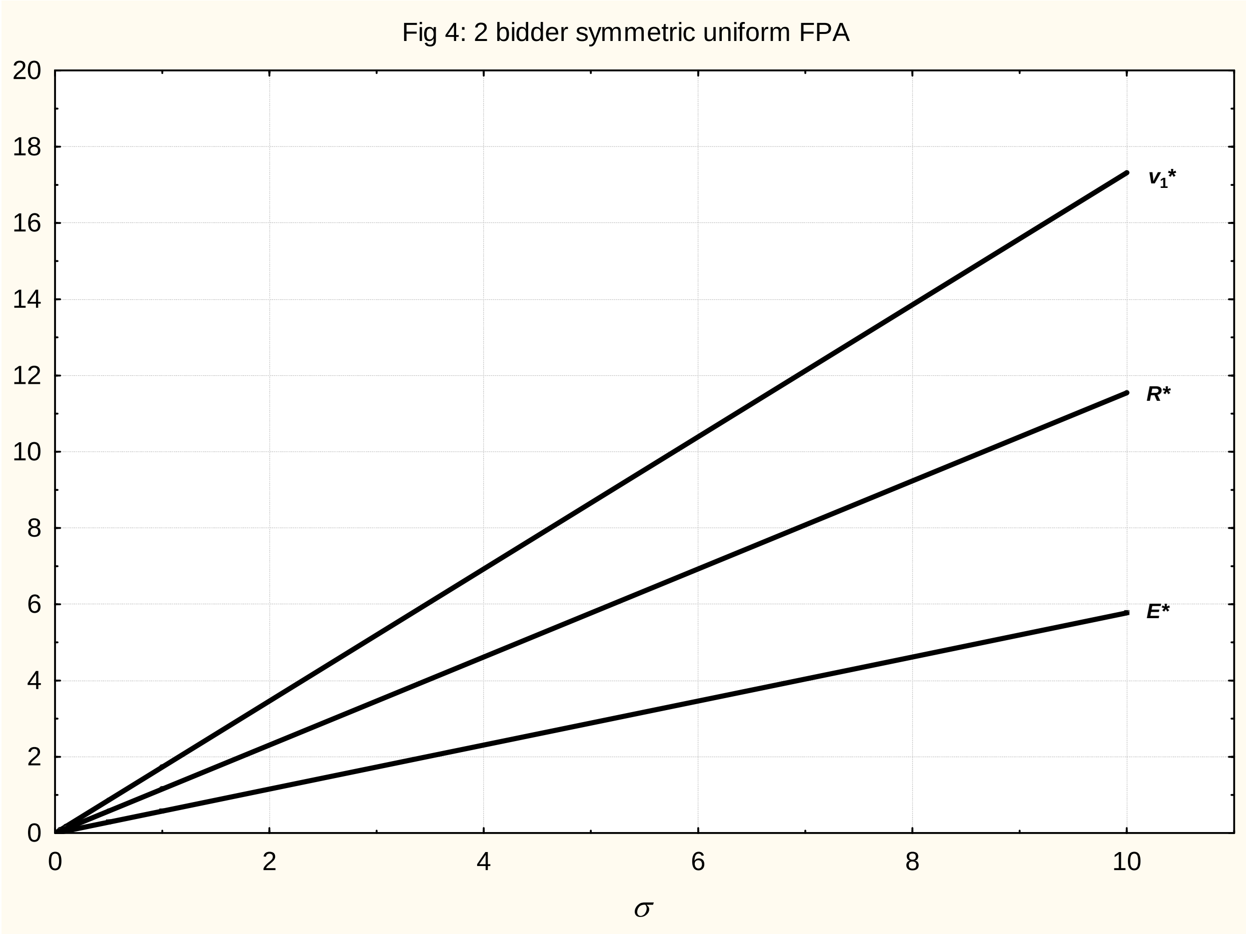


Fig 4: 2 bidder symmetric uniform FPA
20
18
16
14
12
10
8
6
4
2
0
0
2
4
6
8
10
σ
v1*
R*
E*

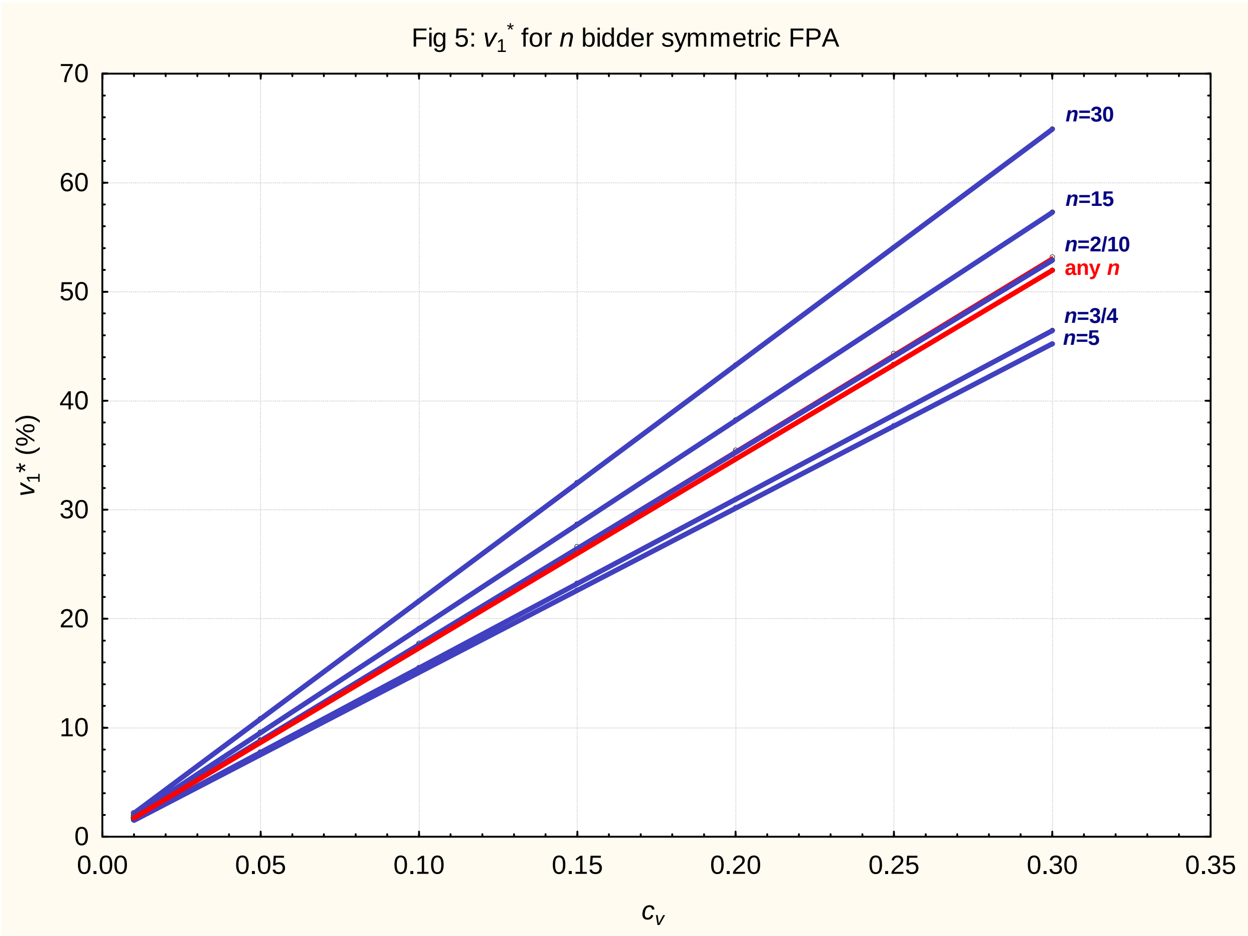


Fig 5: v_1^* for n bidder symmetric FPA
n=30
n=15
n=2/10
any n
n=3/4
n=5
v_1* (%)
c_v
0
10
20
30
40
50
60
70
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35

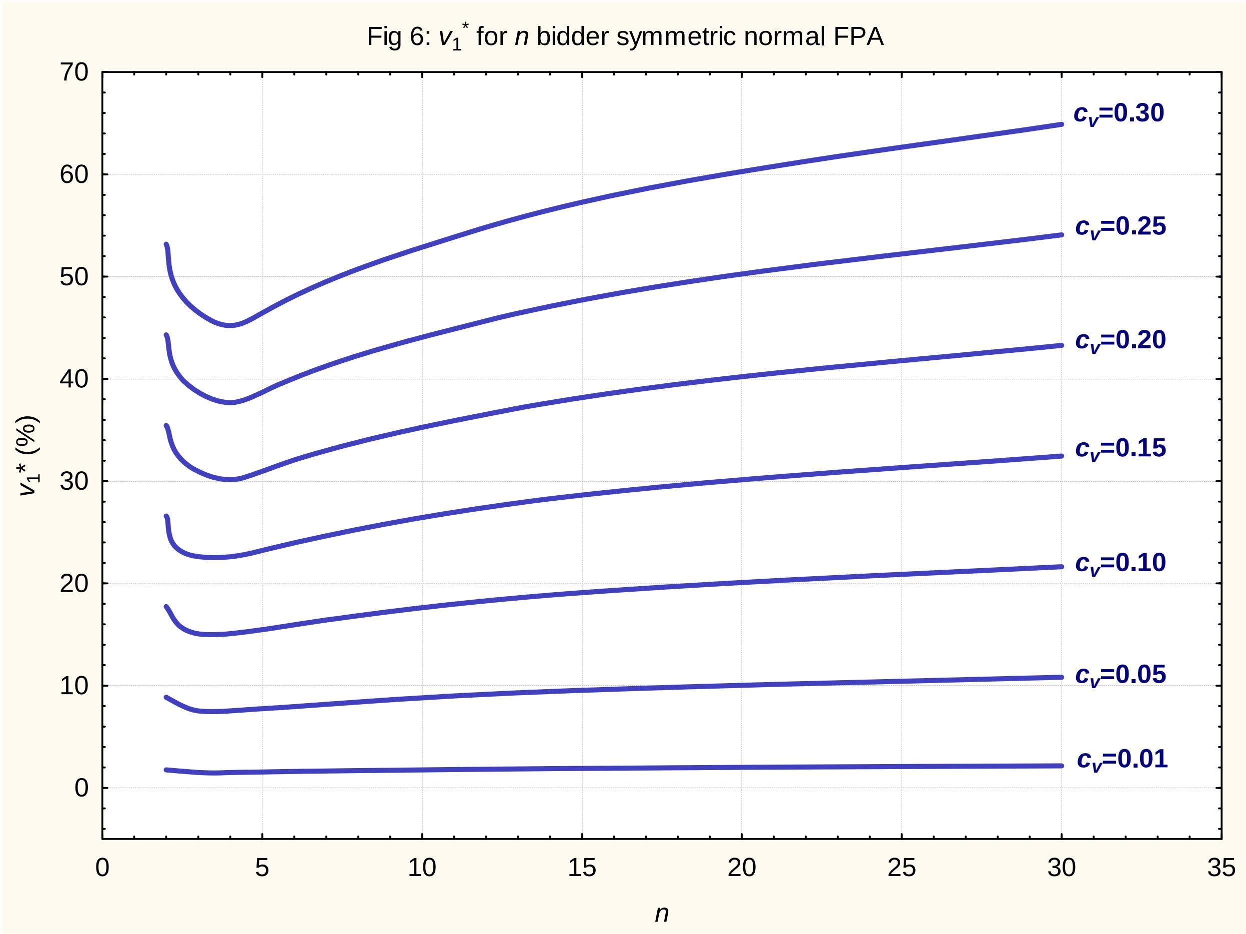


Fig 6: v1* for n bidder symmetric normal FPA
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
n
v1* (%)
cv=0.30
cv=0.25
cv=0.20
cv=0.15
cv=0.10
cv=0.05
cv=0.01

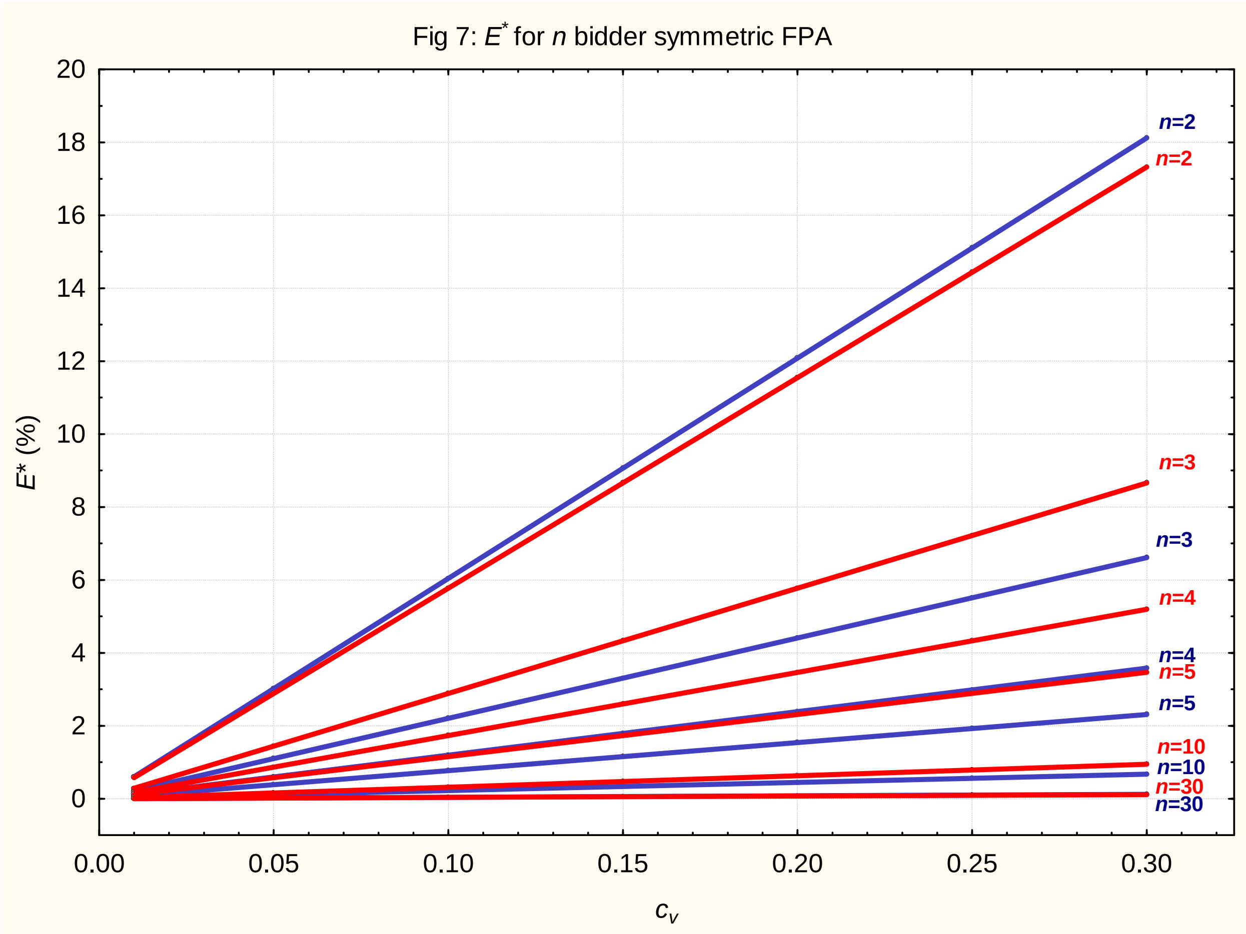


Fig 7: E^* for n bidder symmetric FPA
E^* (%)
c_v
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
2
4
6
8
10
12
14
16
18
20
n=2
n=2
n=3
n=3
n=4
n=4
n=5
n=5
n=10
n=10
n=30
n=30

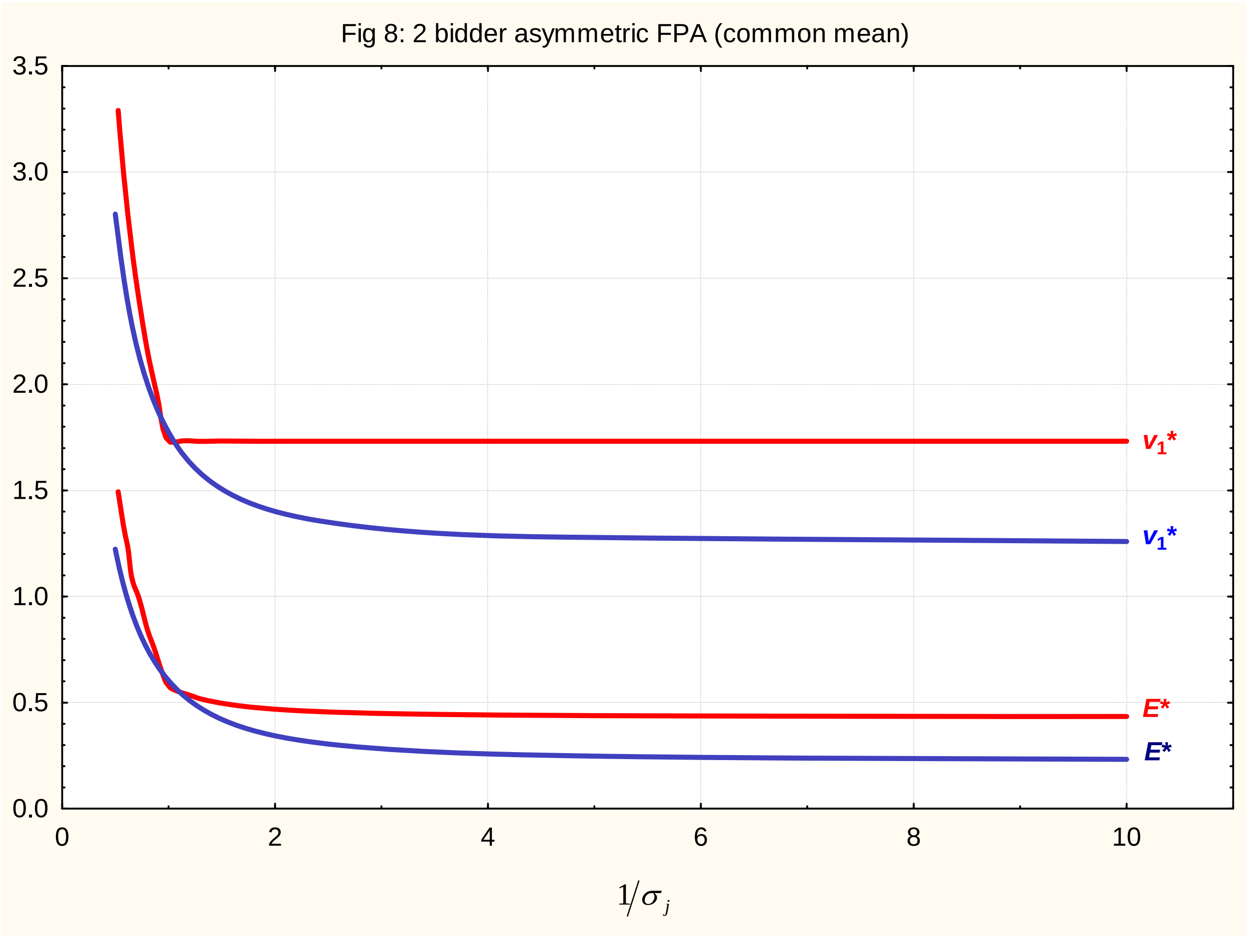


Fig 8: 2 bidder asymmetric FPA (common mean)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
v_1*
v_1*
E*
E*
$1/\sigma_j$

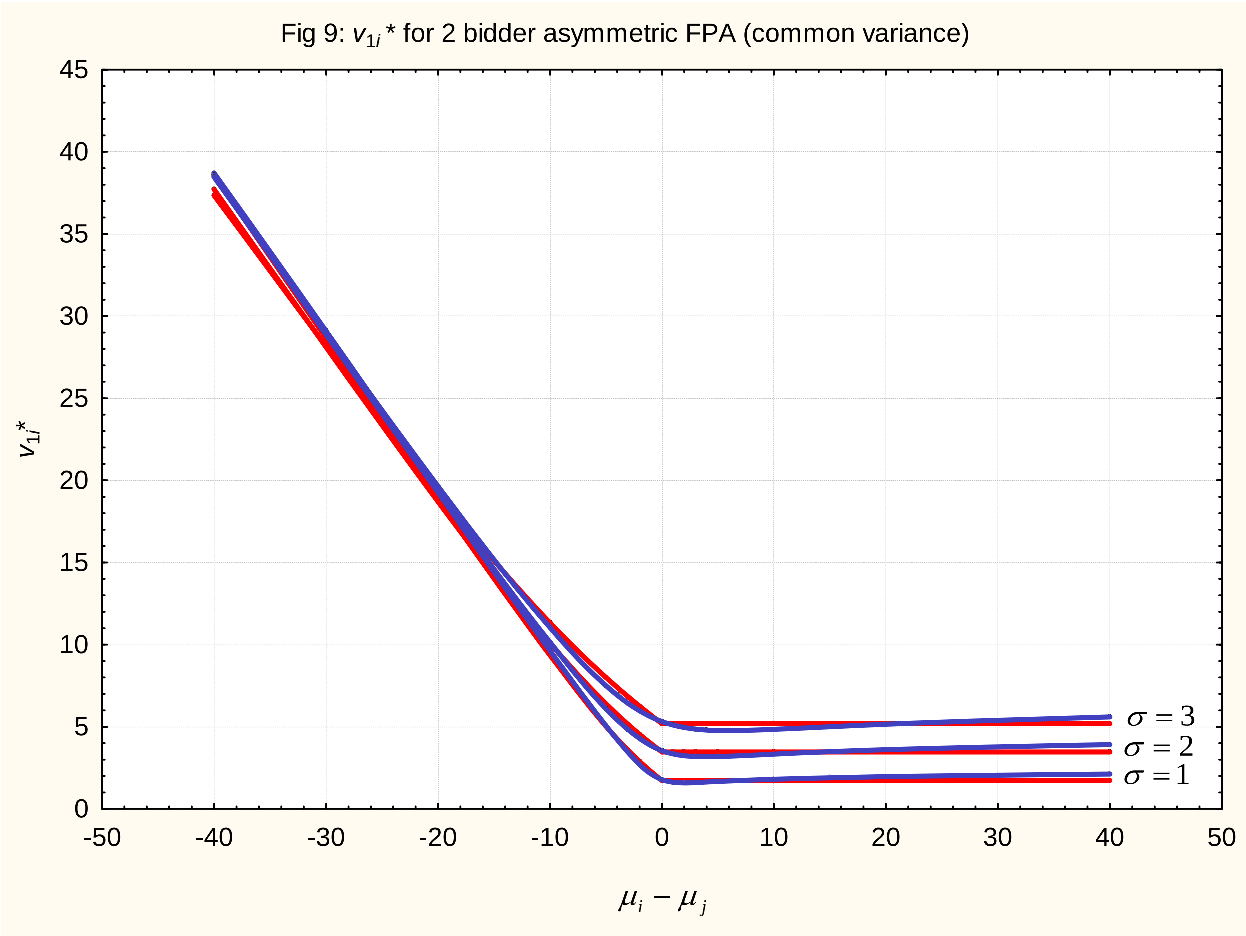


Fig 9: v_{1i}* for 2 bidder asymmetric FPA (common variance)
45
40
35
30
25
20
15
10
5
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
v_{1i}*
$\mu_i - \mu_j$
$\sigma = 3$
$\sigma = 2$
$\sigma = 1$

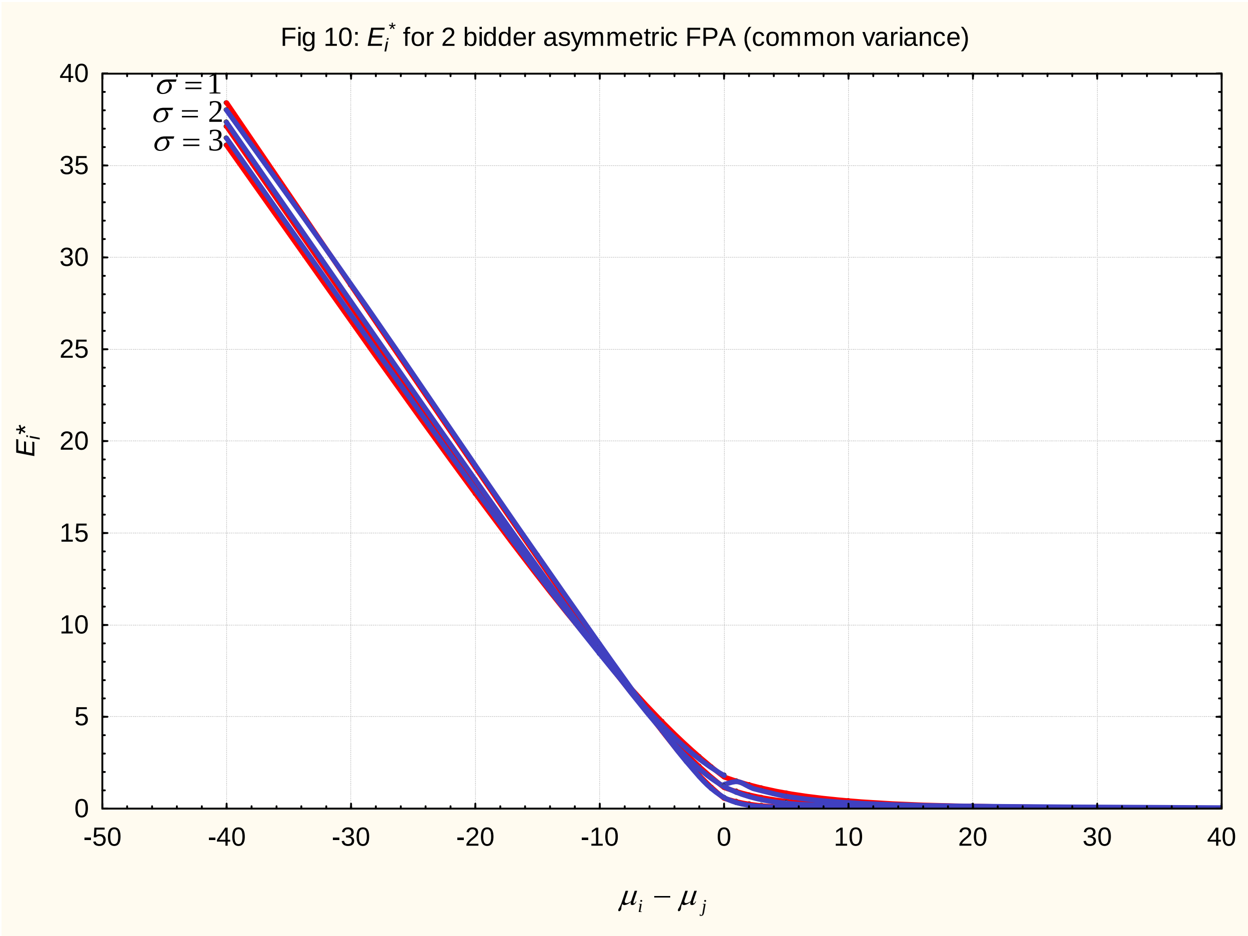


Fig 10: E_i^* for 2 bidder asymmetric FPA (common variance)
$\sigma = 1$
$\sigma = 2$
$\sigma = 3$
E_i^*
40
35
30
25
20
15
10
5
0
-50
-40
-30
-20
-10
0
10
20
30
40
$\mu_i - \mu_j$

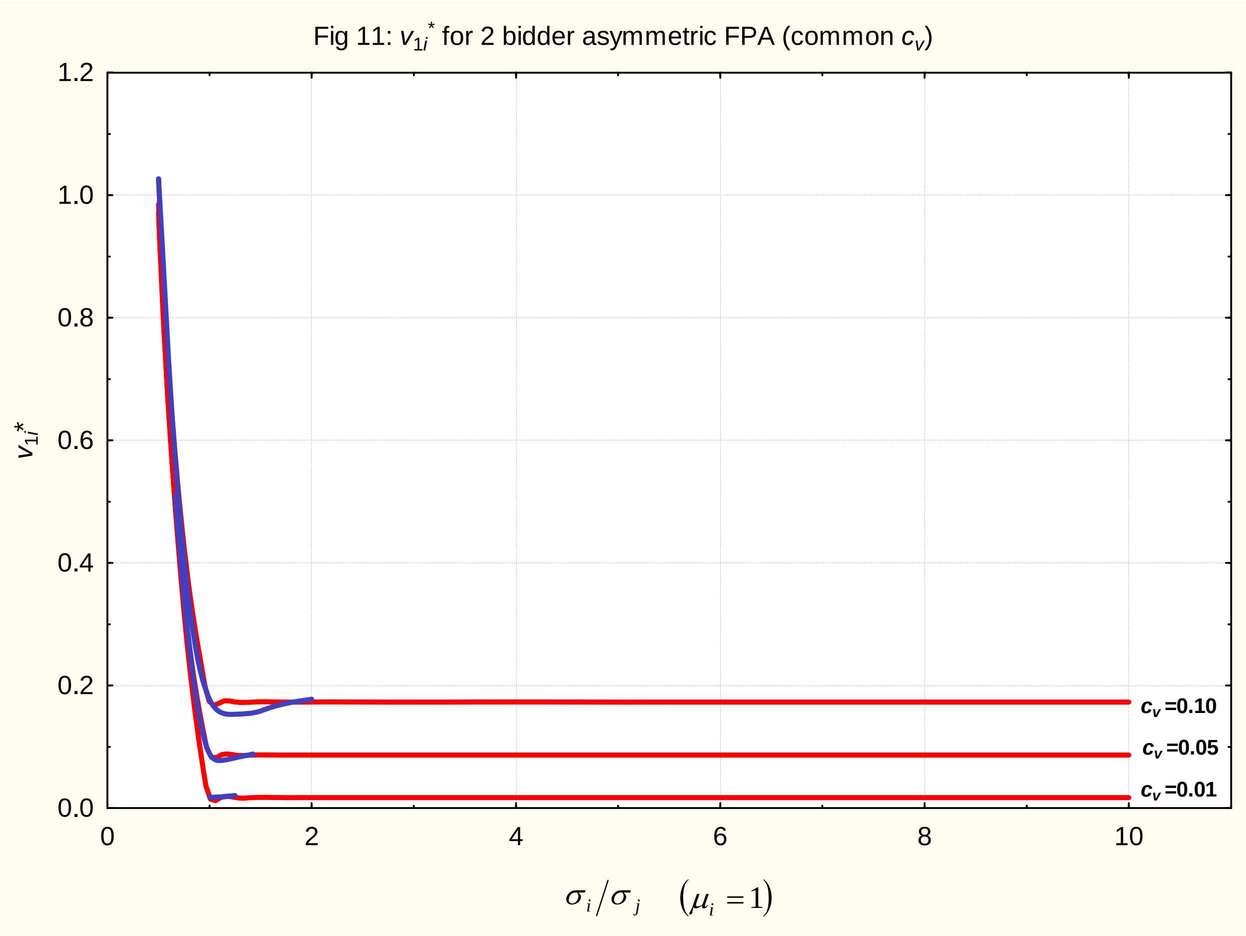


Fig 11: v_{1i}^* for 2 bidder asymmetric FPA (common c_v)
v_{1i}^*
$\sigma_i/\sigma_j \quad (\mu_i = 1)$
c_v =0.10
c_v =0.05
c_v =0.01
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
2
4
6
8
10

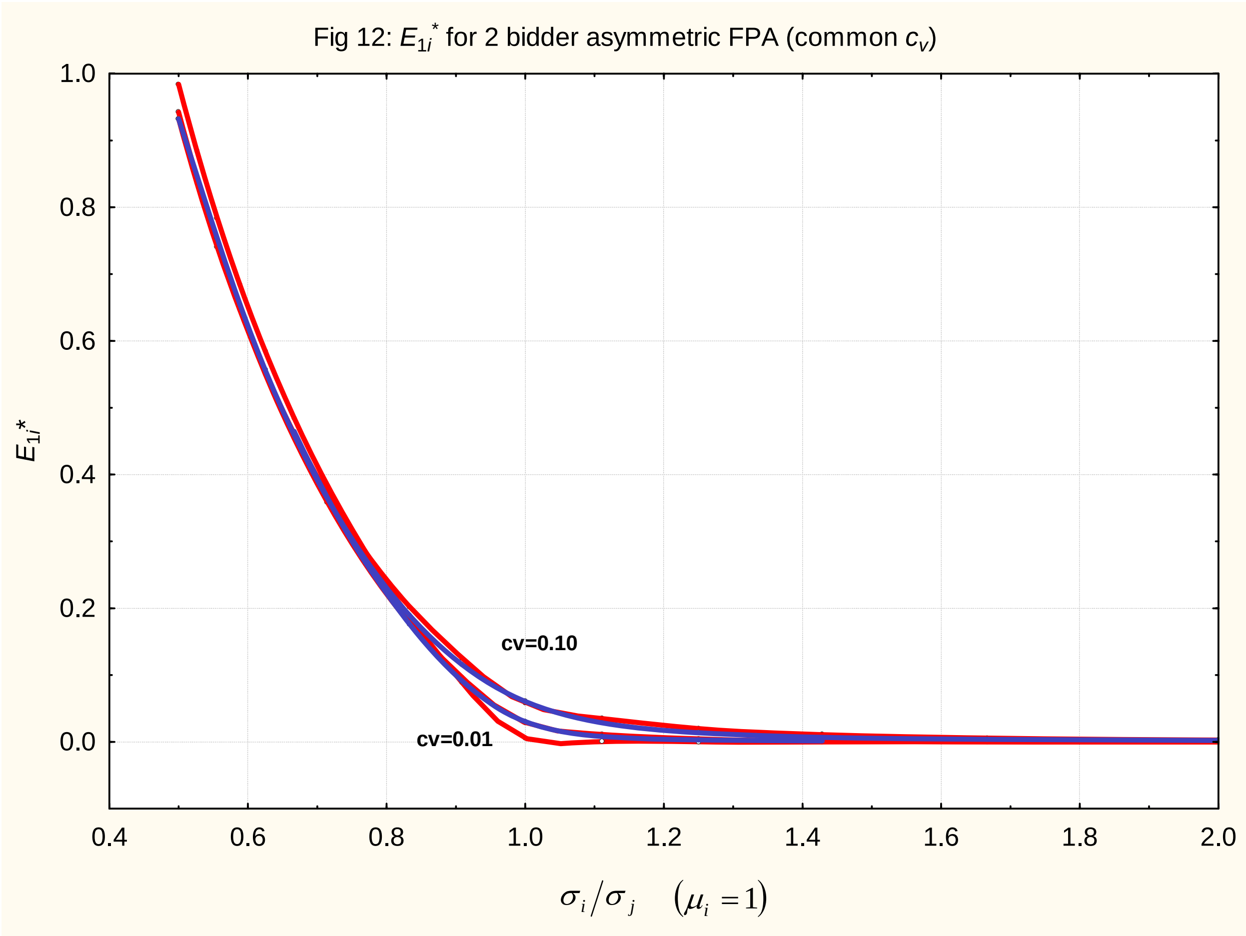


Fig 12: E_{1i}^* for 2 bidder asymmetric FPA (common c_v)
E_{1i}^*
1.0
0.8
0.6
0.4
0.2
0.0
cv=0.10
cv=0.01
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
$\sigma_i / \sigma_j \quad (\mu_i = 1)$

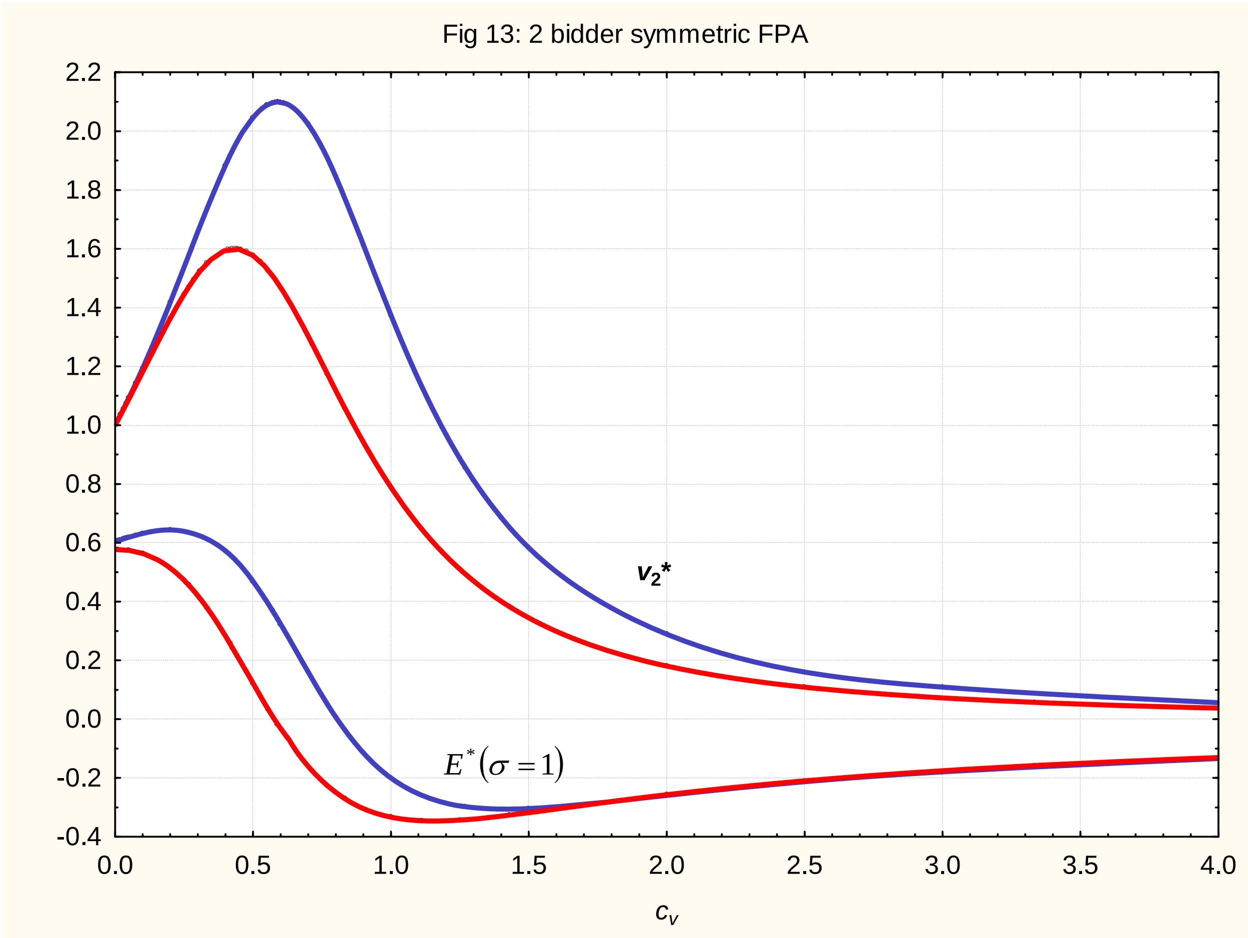
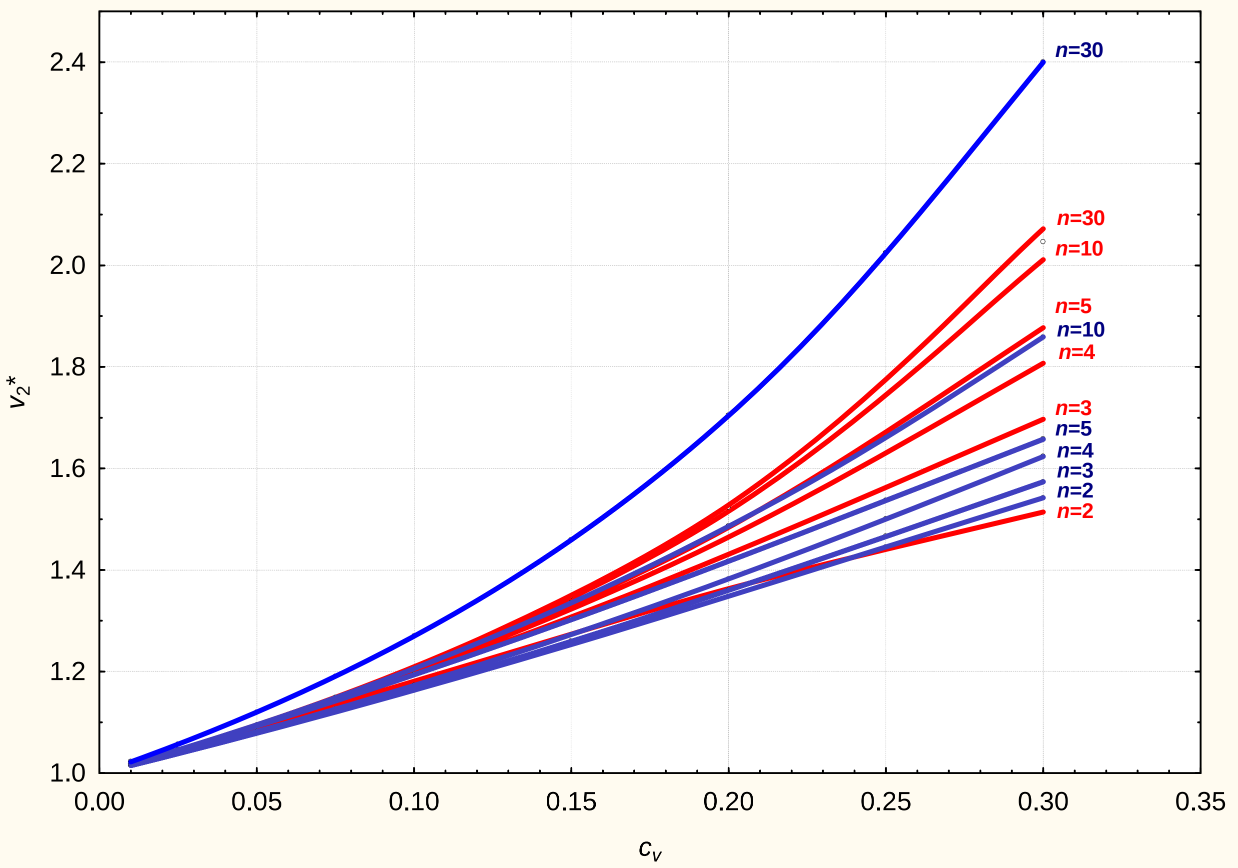


Fig 13: 2 bidder symmetric FPA
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
c_v
v_2*
$E^*(\sigma=1)$

Fig 14: v_2* for n bidder symmetric FPA



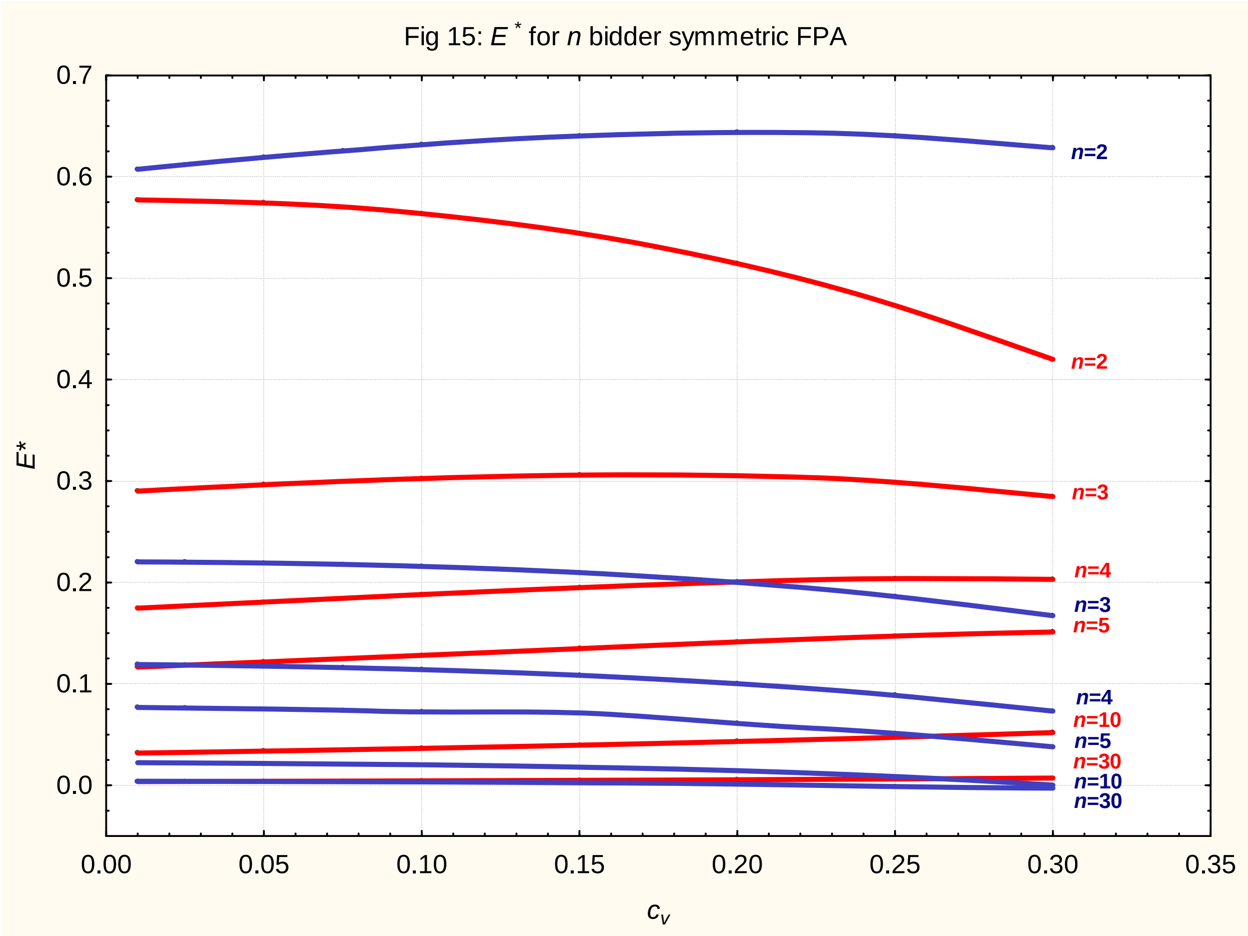


Fig 15: E^* for n bidder symmetric FPA
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
E^*
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
c_v
n=2
n=2
n=3
n=4
n=3
n=5
n=4
n=10
n=5
n=30
n=10
n=30

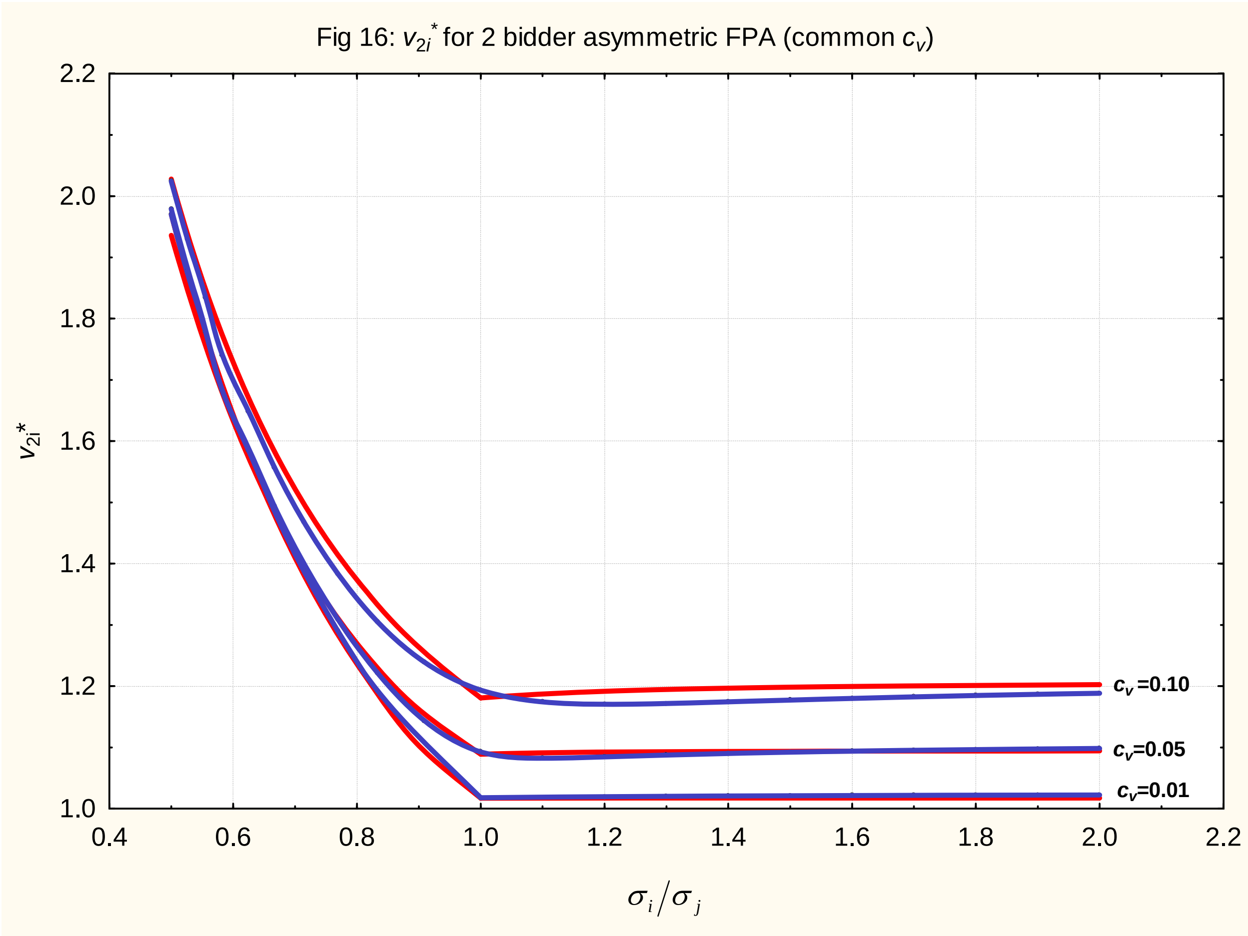


Fig 16: v_{2i}^{*} for 2 bidder asymmetric FPA (common c_v)
v_{2i}^{*}
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
c_v=0.10
c_v=0.05
c_v=0.01
σ_i/σ_j